# A POSTERIORI ERROR ESTIMATES ON ANISOTROPIC MESHES

Natalia Kopteva

*University of Limerick, Ireland*

***II International conference***
***"Multiscale methods and Large-scale Scientific Computing"***
***INM RAS, Moscow, August 2018***

- For singularly perturbed *semilinear reaction-diffusion* equations

$$-\varepsilon^2 \triangle u + f(x,u) = 0$$

where $x \in \Omega \subset \mathbb{R}^2$, subject to $u = 0$ on $\partial\Omega$

$$f(x,u) - f(x,v) \geqslant C_f [u - v] \text{ whenever } u \geqslant v, \quad \boxed{\varepsilon^2 + C_f \gtrsim 1}$$

we look for *residual-type a posteriori error estimates*

$$\boxed{\|\text{error}\|_* \leqslant \text{function}\big(\text{mesh, comp.sol-n}\big)}$$

where $\|\cdot\|_*$ is the ***maximum norm*** or the ***energy norm***

on **anisotropic meshes**

- To simplify the presentation, main focus will be on $\boxed{\varepsilon = 1}$ (so the energy norm becomes the $H^1$ norm); but similar results for $\varepsilon \ll 1$...

Why anisotropic meshes?

**Section A**

**Perceptions & expectations t.b. adjusted** for anisotropic meshes

**Section B**

Part 0 — Standard residual-type estimators on shape-regular meshes; their relation to interpolation errors

Part 1 — **Recent a posteriori estimates on anisotropic meshes**

Part 2 — **A bit of analysis**: 3 technical issues addressed...

Part 3 — **Some Numerics**

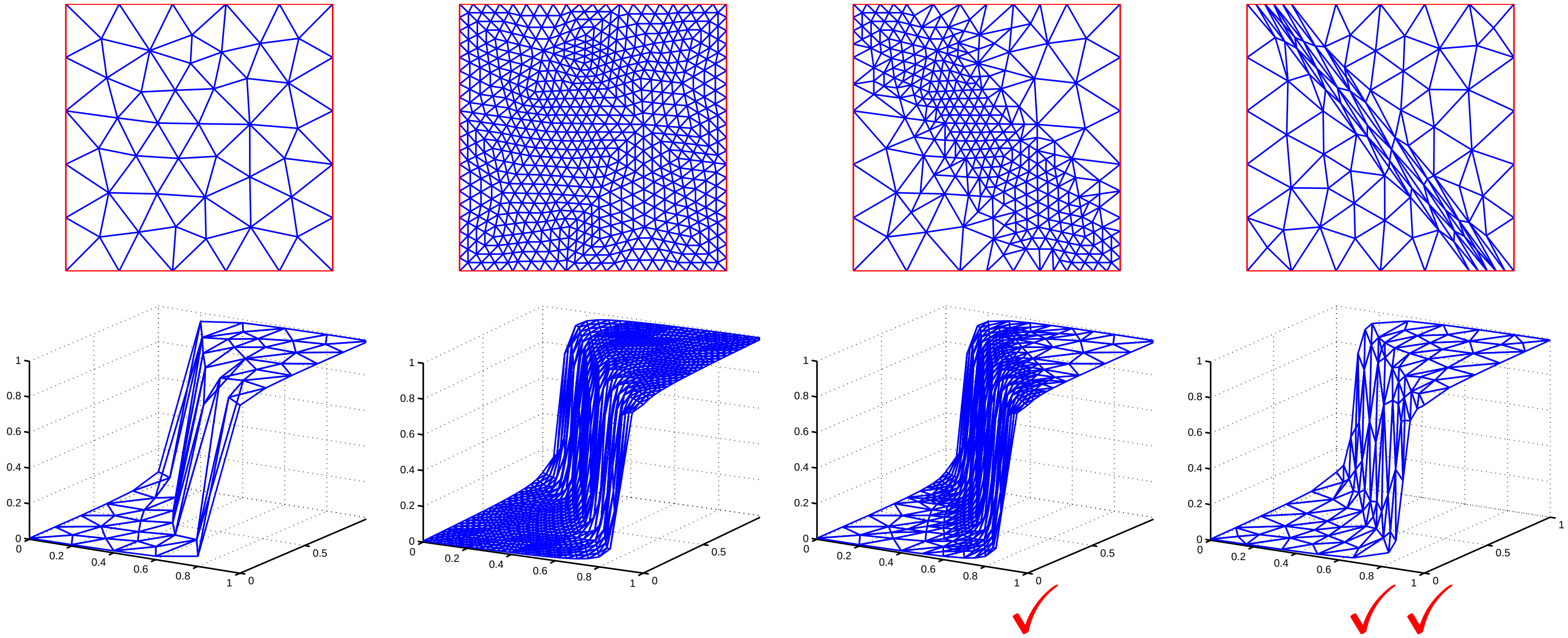
**Section C**

**Efficiency, i.e. lower estimator**: also problematic on anisotropic meshes...

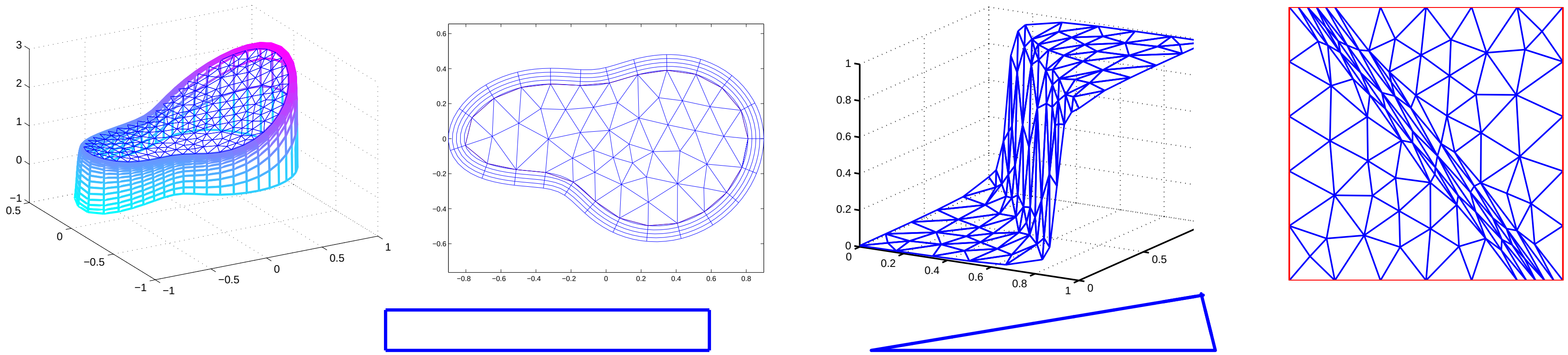
- Interpolation error bounds $\Rightarrow$

**anisotropic meshes are superior for layer solutions**

(a) Standard mesh. (b) Fine mesh. (c) Shape-regular refinement. (d) *Anisotropic* ref-nt.



- anisotropic meshes are superior for layer solutions



BUT **theoretical difficulties within the FEM framework...**

– It's not just about working hard and tracking all the constants very carefully

– New tricks are required...

ALSO **Perceptions and expectations t.b. adjusted for anisotropic meshes**

One Perception: **the computed-solution error in the maximum norm is closely related to the corresponding interpolation error...**

- Quasi-uniform meshes, linear elements

$$\|u - u_h\|_{L_\infty(\Omega)} \leqslant \ln(C + \varepsilon/h) \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty(\Omega)}$$

  – Schatz, Wahlbin, *On the quasi-optimality in $L_\infty$ of the $\overset{\circ}{H}{}^1$-projection into finite element spaces*, Math. Comp. 1982: $-\triangle u = f$,

  – Schatz, Wahlbin, *On the finite element method for singularly perturbed reaction-diffusion problems* ..., Math. Comp., 1983: $-\varepsilon^2 \triangle u + au = f$, One Perception: **the computed-solution error in the maximum norm is closely related to the corresponding interpolation error...**

- Quasi-uniform meshes, linear elements

$$\|u - u_h\|_{L_\infty(\Omega)} \leqslant \ln(C + \varepsilon/h) \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty(\Omega)}$$

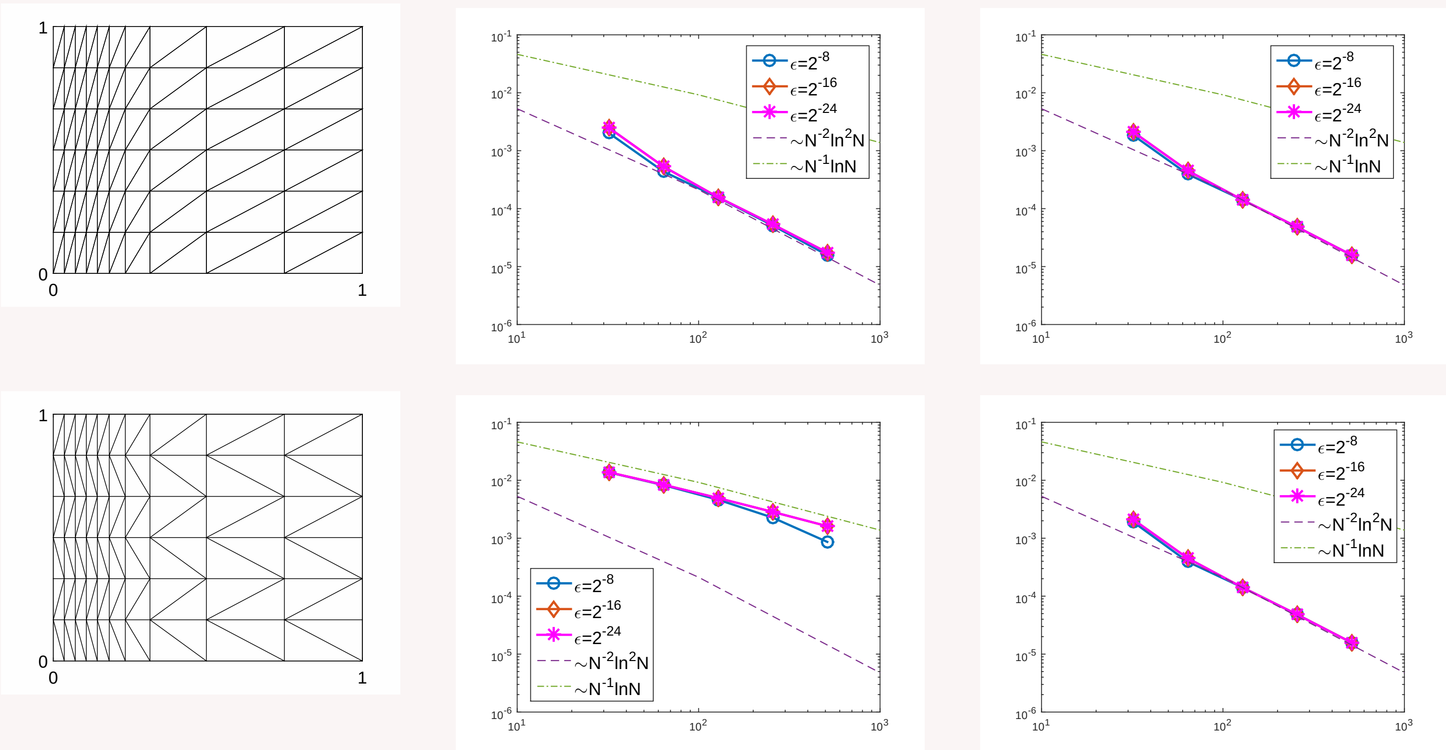
  - Schatz, Wahlbin, *On the quasi-optimality in $L_\infty$ of the $\overset{\circ}{H}{}^1$-projection into finite element spaces*, Math. Comp. 1982: $-\triangle u = f$,
  - Schatz, Wahlbin, *On the finite element method for singularly perturbed reaction-diffusion problems* ..., Math. Comp., 1983: $-\varepsilon^2 \triangle u + au = f$,

- **Strongly-anisotropic triangulations:** no such result
  - BUT this is frequently considered a reasonable heuristic conjecture t.b. used in the anisotropic mesh adaptation (Hessian-related metrics...)
  - IN FACT, this is **NOT true** (see next)

Example: $-\varepsilon^2 \triangle u + u = 0$ with $u = e^{-x/\varepsilon}$ exhibiting a sharp boundary layer

Observation #1: **Mass Lumping may be superior on anisotropic meshes**

Standard linear FEM | Mass Lumping



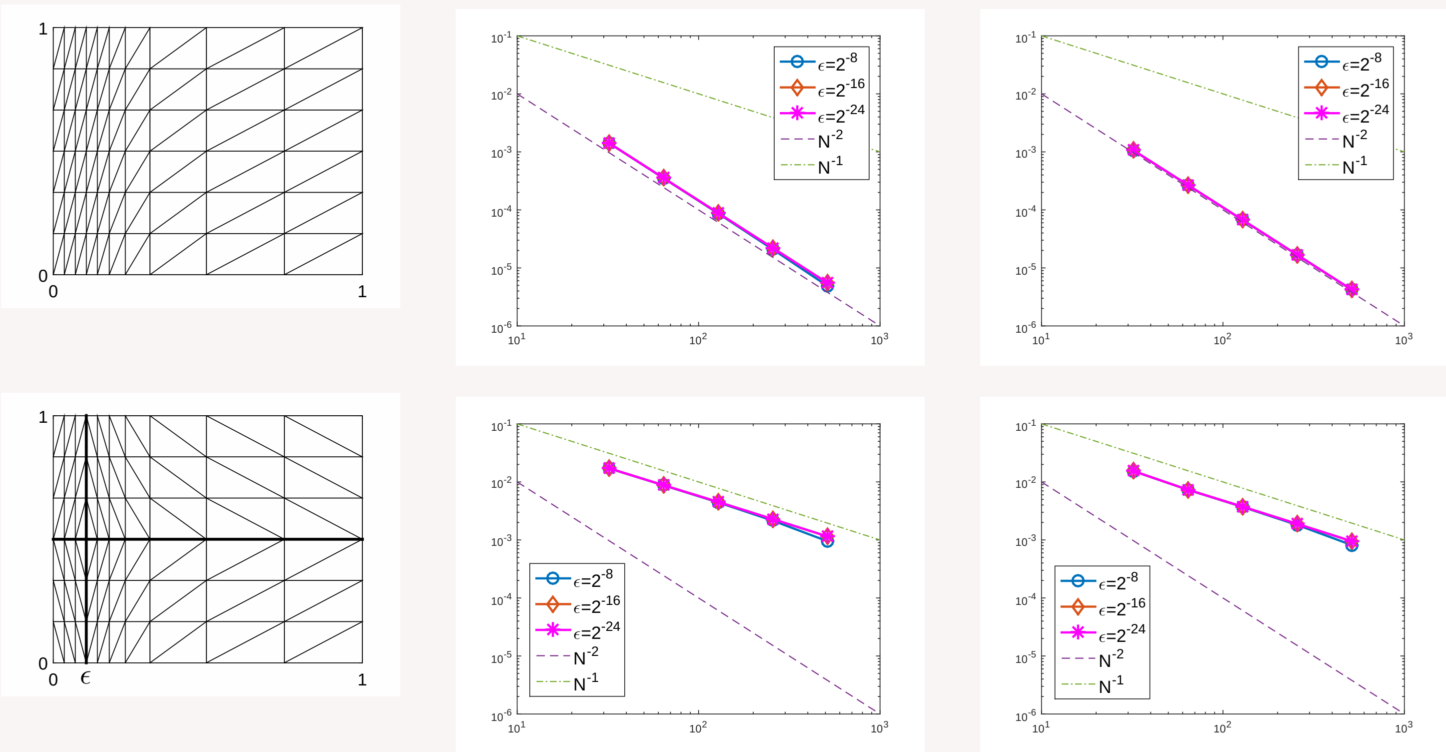

Here we use a Shishkin mesh: piecewise-uniform, $DOF \simeq N^2$, mesh diameter $\simeq N^{-1}$

$$\|u - u^I\|_{L_\infty(\Omega)} \simeq N^{-2} \ln^2 N \simeq DOF^{-1} \ln(DOF)$$

Same Example: $-\varepsilon^2 \triangle u + u = 0$ with $u = e^{-x/\varepsilon}$ exhibiting a **sharp boundary layer**

Observation #2: **Convergence Rates may depend on the mesh structure** (even for mass lumping), **NOT ONLY on the interpolation error**

Standard linear FEM          Mass Lumping




Here we use a graded Bakhvalov mesh:

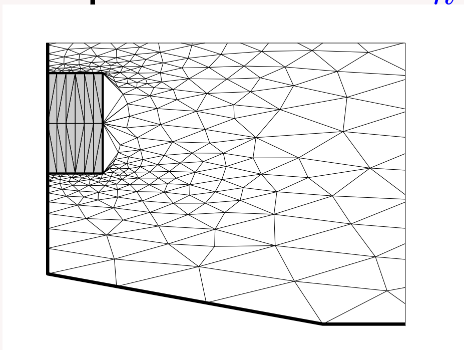
$$\|u - u^I\|_{L_\infty(\Omega)} \simeq N^{-2} \simeq DOF^{-1}$$

- A theoretical explanation of the above phenomena is given in:

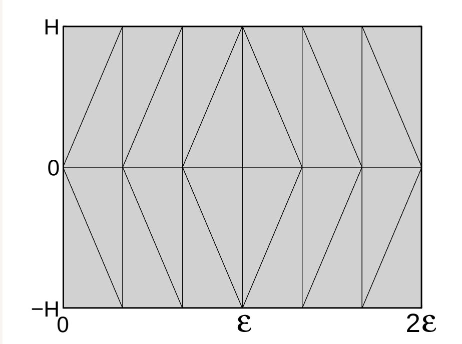
  N.Kopteva, *Linear finite elements may be only first-order pointwise accurate on anisotropic triangulations*, Math. Comp., 2014.

What happens in $\mathring{\Omega} := (0, 2\varepsilon) \times (-H, H)$
with the tensor-product mesh $\mathring{\omega}_h := \{x_i = \varepsilon \frac{i}{N_0}\}_{i=0}^{2N_0} \times \{-H, 0, H\}$??

$\mathcal{T}$ in $\Omega$:



$\mathcal{T}_0$ in $\Omega_0 \subset \Omega$:




Mass lumping, $U_i := u_h(x_i, 0)$ and $U_i^{\pm} := u_h(x_i, \pm H)$:

$$\frac{\varepsilon^2}{h^2}[-U_{i-1} + 2U_i - U_{i+1}] + \frac{\varepsilon^2}{H^2}[-U_i^- + 2U_i - U_i^+] + \gamma_i U_i = 0$$

with $\gamma_i = 1$ for $i \neq N_0$, and $\gamma_{N_0} = \frac{2}{3}$

$$\varepsilon \ll \mathbf{H} \quad \Rightarrow \quad \frac{\varepsilon^2}{h^2}[-U_{i-1} + 2U_i - U_{i+1}] + \frac{\varepsilon^2}{H^2}[-U_i^- + 2U_i - U_i^+] + \gamma_i U_i = 0$$

Implications of the above example:

- Theoretical:

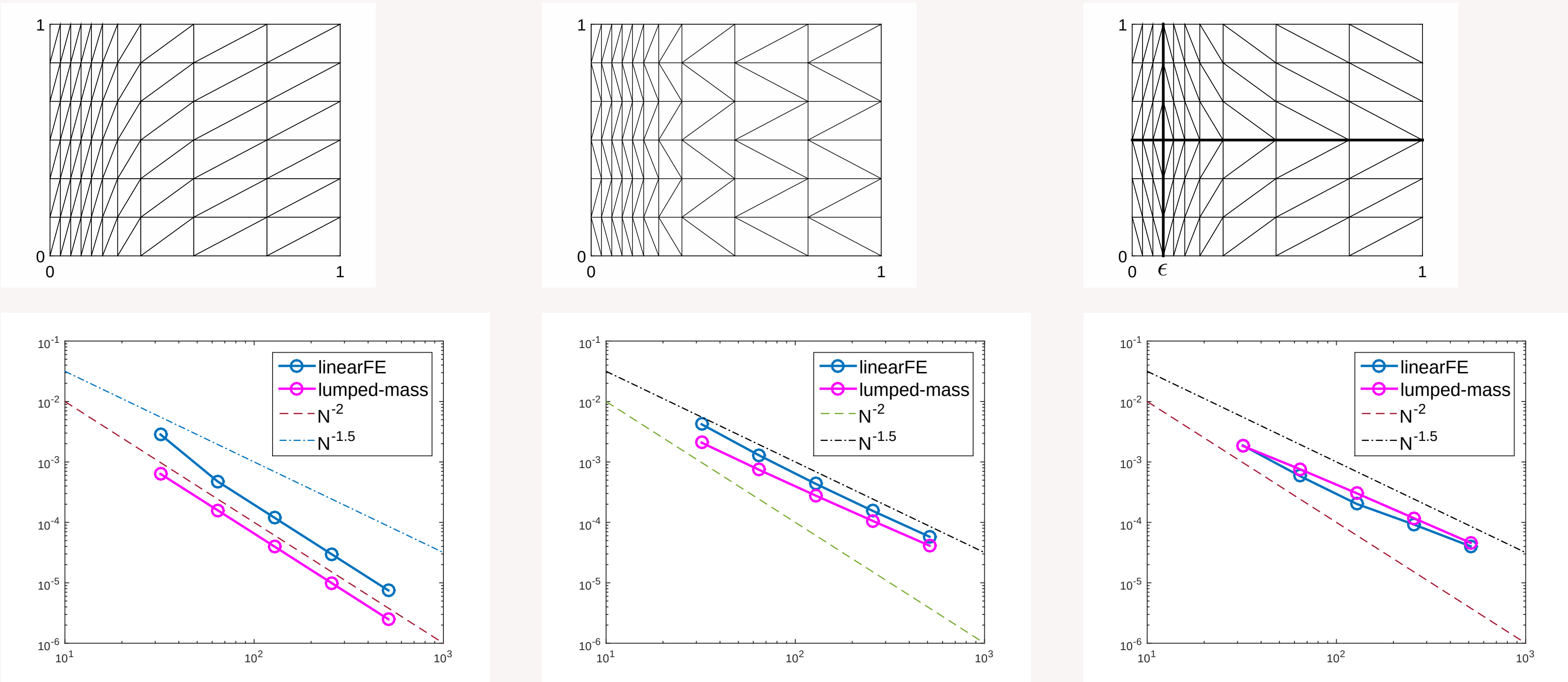
  if one tries to prove ”standard” (almost) second-order a priori/a posteriori error estimate in the maximum norm on a general anisotropic mesh, this may be impossible...

- Anisotropic mesh adaptation (Hessian-related metrics...):

  One needs to be careful with the heuristic conjecture that the computed-solution error in the maximum norm is closely related to the corresponding interpolation error...

Non-singularly-perturbed EXAMPLE [Nochetto et al, Numer. Math., 2006]:

$$-\triangle u + f(u) = 0 \text{ with } f(u) \sim -u^{-3} \text{ and } u = \sqrt{x}$$




Graded mesh: $\{(i/N)^6\}_{i=0}^N$:

$$\|u - u^I\|_{L_\infty(\Omega)} \simeq N^{-2} \simeq DOF^{-1}$$

Mesh transition parameter: $\epsilon = 0.1$

Why anisotropic meshes?

**Section A**

**Perceptions & expectations t.b. adjusted** for anisotropic meshes

**Section B**

Part 0 — Standard residual-type estimators on shape-regular meshes; their relation to interpolation errors

Part 1 — **Recent a posteriori estimates on anisotropic meshes**

Part 2 — **A bit of analysis**: 3 technical issues addressed...

Part 3 — **Some Numerics**

**Section C**

**Efficiency, i.e. lower estimator**: also problematic on anisotropic meshes...

**Laplace equation** $-\triangle u = f(x)$, linear elements, shape-regular mesh

[Ainsworth & Oden, 2000, Chap. 2]

- $H^1$ norm [Babuška & Miller, 1987]

$$\|u_h - u\|_{H^1(\Omega)} \lesssim \Big\{ \sum_{T\in\mathcal{T}} \Big( \underbrace{\|h_T\, f\|^2_{L_2(T)}}_{\sim\|h_T \triangle u\|^2_{L_2(T)}} + \underbrace{h_T^2\, \|[\![\nabla u_h]\!]\|^2_{L_\infty(\partial T)}}_{\sim\|h_T\, D^2 u\|^2_{L_2(T)}} \Big) \Big\}^{1/2}$$

$$\sim \|h_T\, D^2 u\|_{L_2(\Omega)} \sim \|\text{linear interpolation error}\|_{H^1(\Omega)}$$

---

**Laplace equation** $-\triangle u = f(x)$, linear elements, shape-regular mesh

[Ainsworth & Oden, 2000, Chap. 2]

- $H^1$ norm [Babuška & Miller, 1987]

$$\|u_h - u\|_{H^1(\Omega)} \lesssim \Big\{ \sum_{T\in\mathcal{T}} \Big( \underbrace{\|h_T\, f\|^2_{L_2(T)}}_{\sim \|h_T \triangle u\|^2_{L_2(T)}} + \underbrace{h_T^2\, \|[\![\nabla u_h]\!]\|^2_{L_\infty(\partial T)}}_{\sim \|h_T\, D^2 u\|^2_{L_2(T)}} \Big) \Big\}^{1/2}$$

$$\sim \|h_T\, D^2 u\|_{L_2(\Omega)} \sim \|\text{linear interpolation error}\|_{H^1(\Omega)}$$

---

- $L_\infty$ norm [Eriksson, 1994], [Nochetto, 1995]

$$\|u_h - u\|_{L_\infty(\Omega)} \lesssim \ln(h_{\min}^{-1}) \max_{T\in\mathcal{T}} \Big\{ \underbrace{h_T^2\, \|f\|_{L_\infty(T)}}_{\sim h_T^2\, \|\triangle u\|_{L_\infty(T)}} + \underbrace{h_T\, \|[\![\nabla u_h]\!]\|_{L_\infty(\partial T)}}_{\sim h_T^2\, \|\, D^2 u\|_{L_\infty(T)}} \Big\}$$

$$\sim \|h_T^2 D^2 u\|_{L_\infty(\Omega)} \sim \|\text{linear interpolation error}\|_{L_\infty(\Omega)}$$

**Laplace equation** $-\triangle u = f(x)$, linear elements, shape-regular mesh :

- In the $H^1$ and $L_\infty$ norms:

$$\|\text{error}\|_* \leqslant \text{function}\big(\text{mesh, comp.solution}\big)$$

$$\sim \|\underbrace{\text{linear interpolation error}}_{\text{discrete analogue}}\|_*$$

- Higher-order elements + other norms + other equations have been considered as well.

- PURPOSE of such bounds: to be used in the automatic mesh adaptation...

$-\varepsilon^2\triangle u + f(x,u) = 0$, **shape-regular mesh**, any-order FEM,
also analogous lower bounds...

- **Energy norm** $|||\text{error}|||_{\varepsilon\,;\Omega} := \varepsilon\|\nabla\text{error}\|_{L_2(\Omega)} + \|\text{error}\|_{L_2(\Omega)}$

  [Verfürth, Numer. Math., 1998, $-\varepsilon^2\triangle u + u = f(x)$], for linear FEs :

$$\Big\{\sum_{T\in\mathcal{T}}\Big(\underbrace{\|\min\{1,\tfrac{h_T}{\varepsilon}\}\,f(\cdot,u_h)\|^2_{L_2(T)}}_{\sim\|\varepsilon h_T\triangle u\|^2_{L_2(T)}} + \min\{1,\tfrac{\varepsilon}{h_T}\}\,\underbrace{h_T^2\,\|\varepsilon\,[\![\nabla u_h]\!]\|^2_{L_\infty(\partial T)}}_{\sim\|\varepsilon h_T\,D^2u\|^2_{L_2(T)}}\Big)\Big\}^{1/2}$$

---

$-\varepsilon^2\triangle u + f(x,u) = 0$, **shape-regular mesh**, any-order FEM,
also analogous lower bounds...

- **Energy norm** $|||\text{error}|||_{\varepsilon\,;\Omega} := \varepsilon\|\nabla\text{error}\|_{L_2(\Omega)} + \|\text{error}\|_{L_2(\Omega)}$

  [Verfürth, Numer. Math. 1998, $-\varepsilon^2\triangle u + u = f(x)$], for linear FEs :

$$\Big\{\sum_{T\in\mathcal{T}}\Big(\underbrace{\|\min\{1,\tfrac{h_T}{\varepsilon}\}\, f(\cdot,u_h)\|^2_{L_2(T)}}_{\sim\|\varepsilon h_T\triangle u\|^2_{L_2(T)}} + \min\{1,\tfrac{\varepsilon}{h_T}\}\,\underbrace{h_T^2\,\|\varepsilon\,[\![\nabla u_h]\!]\|^2_{L_\infty(\partial T)}}_{\sim\|\varepsilon h_T\,D^2u\|^2_{L_2(T)}}\Big)\Big\}^{1/2}$$

---

- **$L_\infty$ norm** [Demlow & Kopteva, Numer. Math. 2015], for linear FEs :

$$\max_{T\in\mathcal{T}}\Big\{\min\{1,\ell_h\tfrac{h_T^2}{\varepsilon^2}\}\,\underbrace{\|f(\cdot,u_h)\|_{L_\infty(T)}}_{\sim\varepsilon^2|\triangle_h u_h|+O(h_T^2)} + \min\{\varepsilon,\ell_h h_T\}\,\underbrace{\|[\![\nabla u_h]\!]\|_{L_\infty(\partial T)}}_{\sim h_T|D^2u|}\Big\}$$

where $\ell_h = \ln(2+\varepsilon h_{\min}^{-1})$

$-\varepsilon^2 \triangle u + f(x,u) = 0$, **ANISOTROPIC mesh** :

- **$L_\infty$ norm** [Kopteva, SIAM J. Numer. Anal., 2015, new for $\varepsilon = 1$ and $\varepsilon \ll 1$]

---

$-\varepsilon^2\triangle u + f(x,u) = 0$, **ANISOTROPIC mesh** :

- **$L_\infty$ norm** [Kopteva, SIAM J. Numer. Anal., 2015, new for $\varepsilon = 1$ and $\varepsilon \ll 1$]

---

- **Energy norm** $|||\text{error}|||_{\varepsilon\,;\Omega} = \varepsilon\|\nabla\text{error}\|_{L_2(\Omega)} + \|\text{error}\|_{L_2(\Omega)}$

— [Kunert, Kunert & Verfürth, Numer. Math., 2000, $-\triangle u = f(x)$, $-\varepsilon^2\triangle u + u = f(x)$]

ISSUE: the error constant involves the so-called *matching function* $\boxed{m(u-u_h,\mathcal{T})}$, which may be as large as the mesh aspect ratio $\frac{H_T}{h_T}$,

which is UNDESIRABLE...

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

— [Kopteva, Numer. Math., 2017, new for $\varepsilon = 1$ and $\varepsilon \ll 1$]

extends the framework of [Kopteva, SIAM J. Numer. Anal., 2015]

from the $L_\infty$ to the energy norm... (*NO matching functions!* )

Energy norm

For $\varepsilon = 1$, linear FEM, our ESTIMATOR reduces to

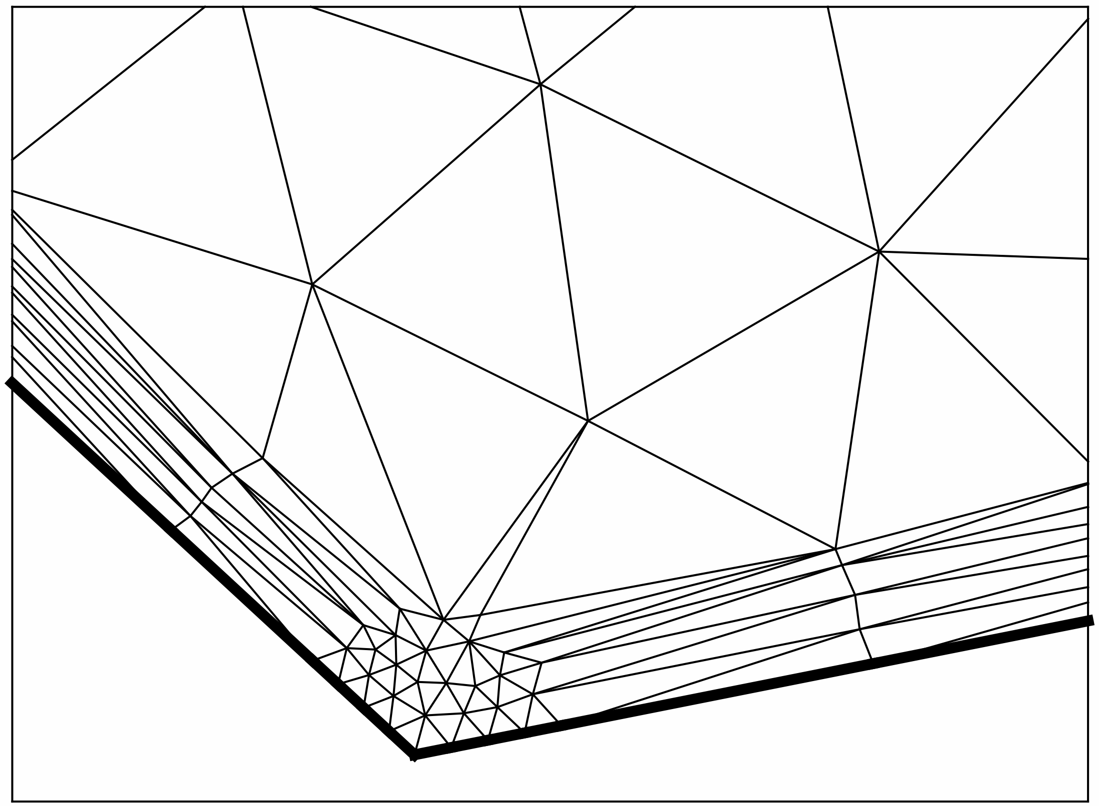
$$\|u_h - u\|_{H^1(\Omega)} \leqslant C \left\{ \sum_{z\in\mathcal{N}} h_z H_z \, \|[\![\nabla u_h]\!]\|^2_{\infty\,;\gamma_z} \right\}^{1/2} + \text{interior-residual terms}$$

**$C$ is independent of the diameters and the aspect ratios of elements in $\mathcal{T}$.**

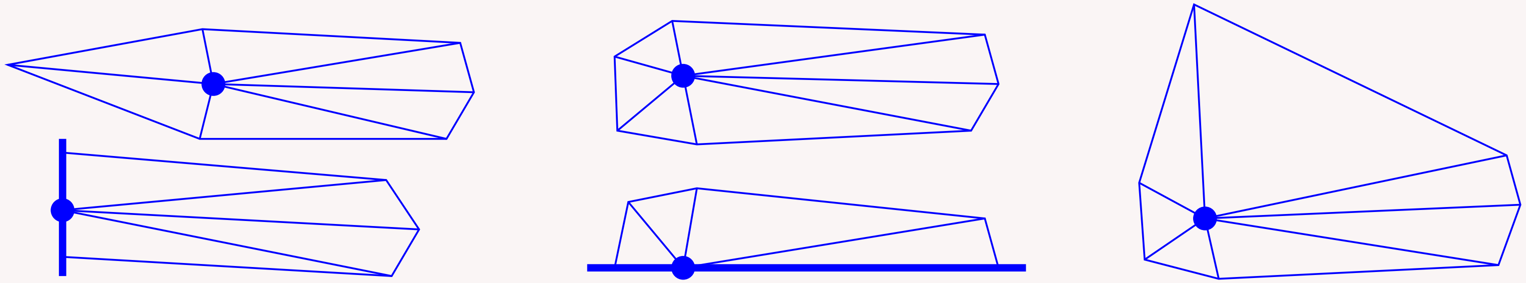
Here $f_h = f(\cdot, u_h)$, $\mathcal{N}$ is the set of nodes in $\mathcal{T}$, $[\![\nabla u_h]\!]$ is the standard jump in the normal derivative of $u_h$ across an element edge, $\omega_z$ is the patch of elements surrounding any $z \in \mathcal{N}$, $\gamma_z$ is the set of edges in the interior of $\omega_z$, $H_z = \mathrm{diam}(\omega_z)$, and $h_z H_z \sim |\omega_z| = \text{local volume}$.

- For $\varepsilon = 1$, this gives a standard a posteriori error bound, similar to [Babuška et al], only now we prove it for anisotropic meshes.
- Relation to interpolation error bounds: $|[\![\nabla u_h]\!]|$ may be interpreted as approximating the diameter of $\omega_z$ under the metric induced by the squared Hessian matrix of the exact solution.

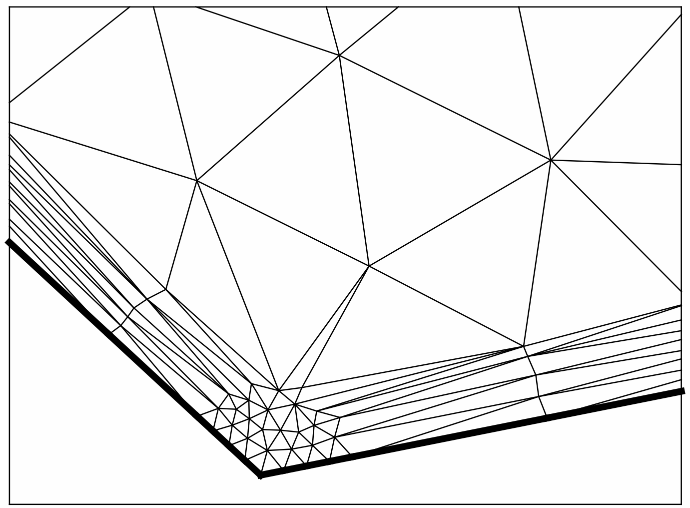
Roughly speaking, want to include meshes of the type:



- Permitted mesh node types:



- Example of a mesh for which the analysis works:

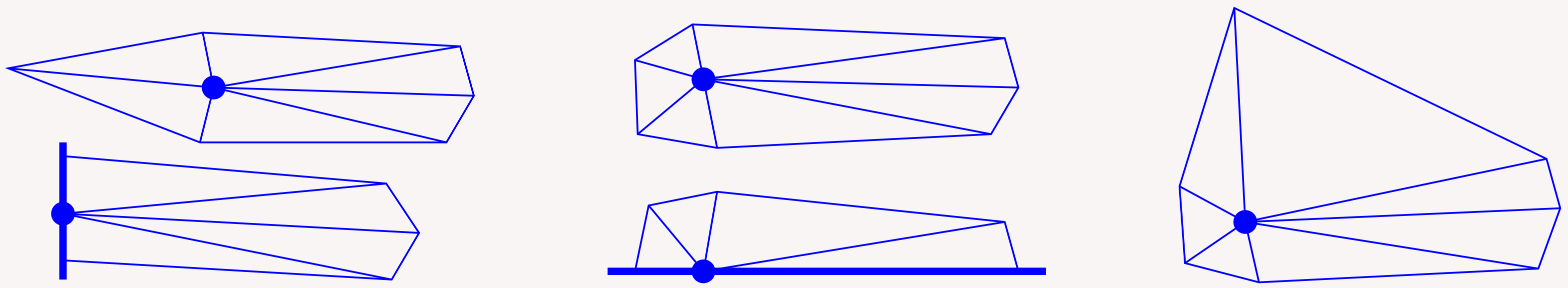


Notation: $H_T := \mathrm{diam}(T)$, $h_T := 2H_T^{-1}|T|$, $H_z := \mathrm{diam}(\omega_z)$, $h_z := \max_{T \subset \omega_z} h_T$

Main Triangulation Assumptions:

- *Maximum Angle condition.*
- *Local Element Orientation condition.* For any $z \in \mathcal{N}$, with the patch $\omega_z$ of elements surrounding $z$, there is a rectangle $R_z \supset \omega_z$ such that $|R_z| \sim |\omega_z|$.
- Also let the number of triangles containing any node be uniformly bounded.

Mesh Node Types:



**Standard Steps:**

— Error represenation via Green's function $G$ ($L_\infty$ norm) or similar (energy norm)

— Use Galerkin orthogonality to replace $G$ by $G - G_h$

— Apply the Divergence Theorem $\Rightarrow$ the error bound includes

Jump Residual terms ($\sum$ integrals over mesh edges)

Interior Residual terms ($\sum$ integrals over mesh elements)

**3 technical issues** t.b. addressed:

1. Application of a Scaled Trace theorem when estimating the Jump Residual (”long” edges cause problems...)
2. Shaper bounds for the Interior Residual (by identifying connected paths of anisotropic nodes...)
3. Quasi-interpolants (of Clément/Scott-Zhang type) are not readily available for general anisotropic meshes [Apel, Chapt. III]...(may be of independent interest)

- For a solution $u$ and any $u_h \in H_0^1(\Omega) \cap W_1^q(\Omega)$ with $q > n = 2$,

$$[u_h - u](x) = \varepsilon^2(\nabla u_h, \nabla G(x,\cdot)) + (f(\cdot, u_h), G(x,\cdot))$$

HINT: using the standard linearization $f(x,u_h) - f(x,u) = p(x)[u_h - u]$

with $p = \int_0^1 f_u(\cdot, u + [u_h - u]s)\, ds \geqslant C_f \geqslant 0$

---

- For each fixed $x \in \Omega$, the Green's function $G = G(x,\cdot)$ solves the problem

$$\begin{aligned} L^*G = -\varepsilon^2\,\Delta_\xi G + p(\xi)\,G &= \delta(x-\xi), & \xi \in \Omega, \\ G(x;\xi) &= 0, & \xi \in \partial\Omega. \end{aligned}$$

(NOTE: similar to the dual problem...)

- For a solution $u$ and any $u_h \in H_0^1(\Omega) \cap W_1^q(\Omega)$ with $q > n = 2$,

$$u_h - u = \varepsilon^2(\nabla u_h, \nabla G) + (f(\cdot, u_h), G)$$

---

- THEOREM [Demlow, Kopteva, 2015] For any $x \in \Omega$,

$$\|G(x,\cdot)\|_{1;\Omega} + \varepsilon\|\nabla G(x,\cdot)\|_{1;\Omega} \lesssim 1.$$

For the ball $B(x,\varrho)$ of radius $\varrho$ centered at $x \in \Omega$, and $\ell_\varrho := \ln(2 + \varepsilon\varrho^{-1})$,

$$\begin{aligned}
\|G(x,\cdot)\|_{1,B(x,\varrho)\cap\Omega} &\lesssim \varepsilon^{-2}\varrho^2\,\ell_\varrho,\\
\|\nabla G(x,\cdot)\|_{1,B(x,\varrho)\cap\Omega} &\lesssim \varepsilon^{-2}\varrho,\\
\|D^2 G(x,\cdot)\|_{1,\Omega\setminus B(x,\varrho)} &\lesssim \varepsilon^{-2}\ell_\varrho
\end{aligned}$$

- For a solution $u$ and any $u_h \in H_0^1(\Omega) \cap W_1^q(\Omega)$ with $q > n = 2$, using the monotonicity of $f$ and $C_f + \varepsilon^2 \geqslant 1$, one gets

$$\begin{aligned}|||u_h - u|||^2_{\varepsilon\,;\Omega} &\lesssim \varepsilon^2 \langle \nabla(u_h - u), \nabla(u_h - u)\rangle + \langle f(\cdot; u_h) - f(\cdot; u), u_h - u\rangle \\ &= \varepsilon^2 \langle \nabla u_h, \nabla(u_h - u)\rangle + \langle f(\cdot; u_h), u_h - u\rangle\,,\end{aligned}$$

where we also used $-\varepsilon^2 \triangle u + f(x, u) = 0$.

Next, assuming $|||u_h - u|||_{\varepsilon\,;\Omega} > 0$, let

$$G := \frac{u_h - u}{|||u_h - u|||_{\varepsilon\,;\Omega}} \qquad \Rightarrow \qquad |||G|||_{\varepsilon\,;\Omega} = 1$$

$$\Rightarrow \quad \boxed{|||u_h - u|||_{\varepsilon\,;\Omega} \lesssim \varepsilon^2 \langle \nabla u_h, \nabla G\rangle + \langle f(\cdot, u_h), G\rangle}$$

— similar to the case of $L_\infty$ norm, only $G$ is no longer the Green's function...

<u>NEXT:</u> $\boxed{\|u_h - u\|_{\dots} = \varepsilon^2(\nabla u_h, \nabla(G - G_h)) + (f_h, G - G_h)}$ $\forall G_h \in S_h$

NEXT: $\|u_h - u\|_{\dots} = \varepsilon^2(\nabla u_h, \nabla(G - G_h)) + (f_h, G - G_h)$ $\forall G_h \in S_h$

NOTE: by the **Divergence Theorem** for each $T \subset \mathcal{T}$,

$$\int_T \nabla u_h \cdot \nabla(G - G_h)) = \int_{\partial T} (G - G_h))\, \nabla u_h \cdot \nu - \int_T \triangle u_h\, (G - G_h))$$

SO

$$\|u_h - u\|_{\dots} = \sum_{S \in \mathcal{S}} \varepsilon^2 \int_S (G - G_h) [\![\nabla u_h]\!] \cdot \nu + \sum_{T \in \mathcal{T}} \int_T (f_h - \varepsilon^2 \underbrace{\triangle u_h}_{=0})\, (G - G_h)$$

NEXT: $\|u_h - u\|_{\dots} = \varepsilon^2(\nabla u_h, \nabla(G - G_h)) + (f_h, G - G_h)$ $\forall G_h \in S_h$

NOTE: by the **Divergence Theorem** for each $T \subset \mathcal{T}$,

$$\int_T \nabla u_h \cdot \nabla(G - G_h)) = \int_{\partial T} (G - G_h))\, \nabla u_h \cdot \nu - \int_T \triangle u_h\, (G - G_h))$$

SO

$$\|u_h - u\|_{\dots} = \sum_{S \in \mathcal{S}} \varepsilon^2 \int_S (G - G_h)[\![\nabla u_h]\!] \cdot \nu + \sum_{T \in \mathcal{T}} \int_T (f_h - \varepsilon^2 \underbrace{\triangle u_h}_{=0})\,(G - G_h)$$

As $\forall G_h \in S_h$, so replace $(G - G_h)$ by

$$G - G_h - \sum_{z \in \mathcal{N}} \bar{g}_z \phi_z = \sum_{z \in \mathcal{N}} [G - G_h - \bar{g}_z]\phi_z$$

where $\phi_z$ = the standard hat function associated with a node $z$

$$\|u_h - u\|_{\dots} = \sum_{z \in \mathcal{N}} \varepsilon^2 \int_{\gamma_z} [G - G_h - \bar{g}_z]\phi_z [\![\nabla u_h]\!] \cdot \nu + \sum_{z \in \mathcal{N}} \int_{\omega_z} f_h\, [G - G_h - \bar{g}_z]\phi_z$$

JUMP RESIDUAL: $$I := \sum_{z\in\mathcal{N}} \varepsilon^2 \int_{\gamma_z} [G - G_h - \bar{g}_z]\phi_z[\![\nabla u_h]\!] \cdot \nu$$ ($\int$ over **edges**)

NOTE: An inspection of standard proofs for shape-regular meshes reveals that one obstacle in extending them to anisotropic meshes lies in the application of a **Scaled Trace Theorem** when estimating the jump residual terms (this causes the mesh aspect ratios to appear in the estimator; **"long" edges** cause this problem).

Scaled Trace Theorem (for anisotropic elements; sharp):

$$\max_{S\in\{\textbf{short edges}\}} \|v\|_{1\,;S} + \frac{\mathbf{h_z}}{\mathbf{H_z}} \max_{S\in\{\textbf{long edges}\}} \|v\|_{1\,;S} \lesssim H_z^{-1}\|v\|_{1\,;\omega_z} + \|\nabla v\|_{1\,;\omega_z}$$

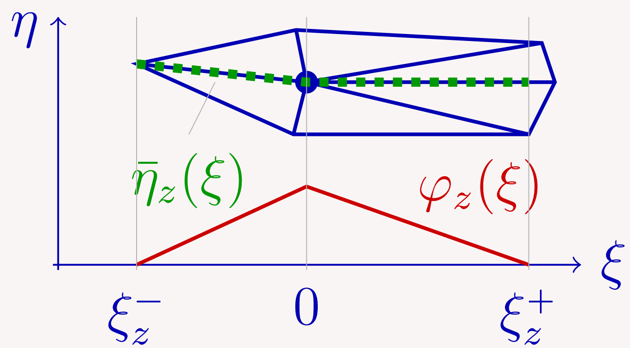
JUMP RESIDUAL: $I := \sum_{z\in\mathcal{N}} \varepsilon^2 \int_{\gamma_z} [G - G_h - \bar{g}_z]\phi_z [\![\nabla u_h]\!] \cdot \nu$

NOTE: An inspection of standard proofs for shape-regular meshes reveals that one obstacle in extending them to anisotropic meshes lies in the application of a **Scaled Trace Theorem** when estimating the jump residual terms (this causes the mesh aspect ratios to appear in the estimator; **”long” edges** cause this problem).

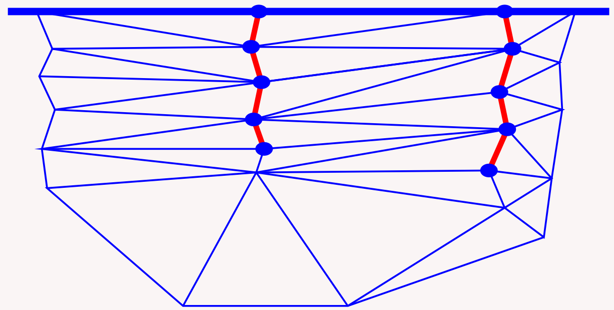
NOTE standard choices: $\bar{g}_z = 0$, or $\int_{\omega_z} (G - G_h - \bar{g}_z)\, \phi_z = 0$ [Nochetto].

Our CHOICE is crucial in addressing this difficulty:

$$\int_{\xi_z^-}^{\xi_z^+} \big[(G - G_h)(\xi, \bar{\eta}_z(\xi)) - \bar{g}_z\big]\, \varphi_z(\xi)\, d\xi = 0$$



In order to give a sharper (and more anisotropic in nature) bound for the interior-residual component of the error, we identify sequences of short edges that connect anisotropic nodes (and call each of them a Path):



Main Additional Assumption: (Curvilinear version also ok...)

- *Path Coordinate-System condition.* For each (semi-)anisotropic path $\mathcal{N}_i$, $i = 1, \ldots, n_{\rm ani} + n_{\rm s.ani}$, let there exist a cartesian coordinate system $(\xi, \eta) = (\xi_i, \eta_i)$ such that $|\sin(\angle(S, \mathrm{i}_\xi))| \lesssim \frac{h_z}{|S|}$ for any $S \subset \mathcal{S}_z$ of any node $z \in \mathcal{N}_i$ (while, if $\mathcal{N}_i$ is semi-anisotropic a stronger condition $|\angle(S, \mathrm{i}_\xi)| \lesssim \frac{h_z}{|S|}$ is satisfied).

TASK: estimate

$$\bar\Theta := \varepsilon^2 \sum_{T\in\mathcal{T}} \Big(\lambda_T^{p-2}\|\nabla(G-G_h)\|_{p\,;T}^p + \lambda_T^{-2}\|G-G_h\|_{p\,;T}^p\Big), \quad \lambda_T := \min\{\varepsilon, H_T\},$$

Aim: $\bar\Theta \lesssim \ell_h$ for $p=1$ for $L_\infty$ norm, or $\bar\Theta \lesssim 1$ for $p=2$...

- It would be convenient to employ a **quasi-interpolant (of Clément/Scott-Zhang type)** with the property

$$|G-G_h|_{k,p\,;T} \lesssim H_T^{j-k}|G|_{j,p\,;\omega_T} \text{ for any } 0 \leqslant \boxed{k \leqslant j} \leqslant 2, \quad p=1.$$

T.b. more precise, the estimator involves $\boxed{\min\{\underbrace{1}_{\text{from } k=j}, \underbrace{\frac{H_T^2}{\varepsilon^2}}_{\text{from } k<j}\}}$

- However, such interpolants are not readily available for general anisotropic meshes (see [Apel, Chapt. III] for a discussion of Scott-Zhang-type interpolation on anisotropic tensor-product meshes).

- It would be convenient to employ a quasi-interpolant (of Clément/Scott-Zhang type) with the property

$$|G - G_h|_{k,p\,;T} \lesssim H_T^{j-k} |G|_{j,p\,;\omega_T} \text{ for any } 0 \leqslant \boxed{k \leqslant j} \leqslant 2,\ \ p = 1.$$

- However, such interpolants are not readily available for anisotropic meshes

- To deal with the maximum norm [Kopteva, 2015]:

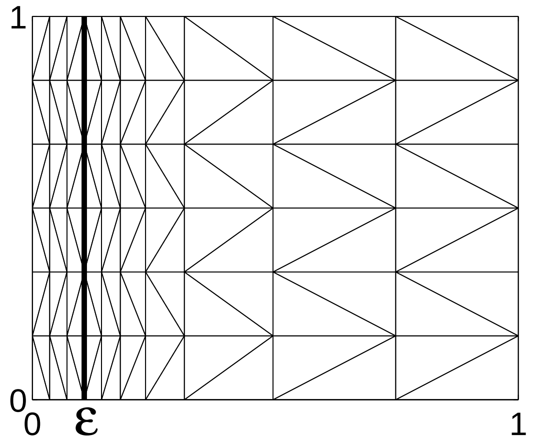
  Because of this difficulty, we employ a less standard interpolant $G_h$, which gives a version of the **Lagrange interpolant** whenever $H_T \lesssim \varepsilon$, and vanishes whenever $H_T \gtrsim \varepsilon$; however, this construction requires additional mild assumptions on the triangulation...

- To deal with the energy norm [Kopteva, 2017]:

  **Quasi-interpolant of Clément/Scott-Zhang type** are introduced on anisotropic meshes...

Simple 2d TEST problem: $-\varepsilon^2 \triangle u + u = F(x)$ in $\Omega = (0,1)^2$ with $\varepsilon^2 = 10^{-6}$,
$u = 4y\,(1-y)\,[1 - x^2 - (e^{-x/\varepsilon} - e^{-1/\varepsilon})/(1 - e^{-x/\varepsilon})]$

We consider one a-priori-chosen layer-adapted mesh of Bakhvalov type:




- The mesh is chosen so that the linear interpolation error $\|u - u^I\|_{\infty\,;\Omega} \lesssim N^{-2}$.
- However, **as $\varepsilon \to 0$, the convergence rates deteriorate from $2$ to $1$**.

  This phenomenon is noted and explained in

  [N. Kopteva, *Linear finite elements may be only first-order pointwise accurate on anisotropic triangulations*, Math. Comp. 2014.].

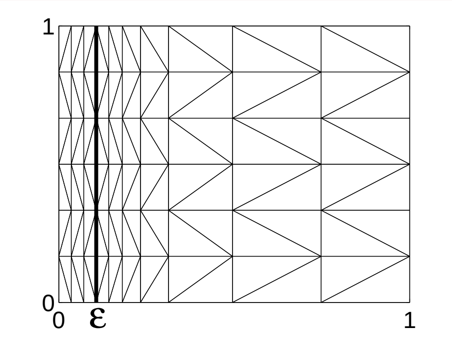
Table: Bakhvalov mesh, $M = \frac{1}{2}N$: **maximum nodal errors** and estimators.

| $N$ | $\varepsilon = 1$ | $\varepsilon = 2^{-5}$ | $\varepsilon = 2^{-10}$ | $\varepsilon = 2^{-15}$ | $\varepsilon = 2^{-20}$ | $\varepsilon = 2^{-25}$ | $\varepsilon = 2^{-30}$ |
|---|---|---|---|---|---|---|---|
| | Errors (odd rows) & Computational Rates (even rows) | | | | | | |
| 64 | 3.373e-4 | 3.723e-3 | 8.952e-3 | 8.973e-3 | 8.973e-3 | 8.973e-3 | 8.973e-3 |
| | 2.00 | 1.91 | 1.01 | 1.00 | 1.00 | 1.00 | 1.00 |
| 128 | 8.445e-5 | 9.935e-4 | 4.446e-3 | 4.484e-3 | 4.484e-3 | 4.484e-3 | 4.484e-3 |
| | 2.00 | 1.98 | 1.04 | 1.00 | 1.00 | 1.00 | 1.00 |
| 256 | 2.112e-5 | 2.523e-4 | 2.165e-3 | 2.236e-3 | 2.236e-3 | 2.236e-3 | 2.236e-3 |
| | FIRST Estimator (odd rows) & **Effectivity Indices** (even rows) | | | | | | |
| 64 | 6.810e-3 | 2.516e-1 | 9.403e-1 | 9.981e-1 | 9.999e-1 | 1.000e+0 | 1.000e+0 |
| | 20.19 | 67.59 | 105.04 | 111.23 | 111.44 | 111.45 | 111.45 |
| 128 | 1.761e-3 | 1.120e-1 | 8.858e-1 | 9.961e-1 | 9.999e-1 | 1.000e+0 | 1.000e+0 |
| | 20.86 | 112.72 | 199.26 | 222.15 | 222.98 | 223.01 | 223.01 |
| 256 | 4.480e-4 | 4.036e-2 | 7.901e-1 | 9.922e-1 | 9.998e-1 | 1.000e+0 | 1.000e+0 |
| | 21.21 | 159.97 | 365.01 | 443.82 | 447.17 | 447.27 | 447.28 |

Table: Bakhvalov mesh, $M = \frac{1}{2}N$: **maximum nodal errors** and estimators.

| $N$ | $\varepsilon = 1$ | $\varepsilon = 2^{-5}$ | $\varepsilon = 2^{-10}$ | $\varepsilon = 2^{-15}$ | $\varepsilon = 2^{-20}$ | $\varepsilon = 2^{-25}$ | $\varepsilon = 2^{-30}$ |
|---|---|---|---|---|---|---|---|
| | Errors (odd rows) & Computational Rates (even rows) | | | | | | |
| 64 | 3.373e-4 | 3.723e-3 | 8.952e-3 | 8.973e-3 | 8.973e-3 | 8.973e-3 | 8.973e-3 |
| | 2.00 | 1.91 | 1.01 | 1.00 | 1.00 | 1.00 | 1.00 |
| 128 | 8.445e-5 | 9.935e-4 | 4.446e-3 | 4.484e-3 | 4.484e-3 | 4.484e-3 | 4.484e-3 |
| | 2.00 | 1.98 | 1.04 | 1.00 | 1.00 | 1.00 | 1.00 |
| 256 | 2.112e-5 | 2.523e-4 | 2.165e-3 | 2.236e-3 | 2.236e-3 | 2.236e-3 | 2.236e-3 |
| | SECOND Estimator (odd rows) & **Effectivity Indices** (even rows) | | | | | | |
| 64 | 7.353e-3 | 1.204e-1 | 1.224e-1 | 1.230e-1 | 1.302e-1 | 1.302e-1 | 1.302e-1 |
| | **21.80** | **32.33** | **13.68** | **14.48** | **14.51** | **14.51** | **14.51** |
| 128 | 1.885e-3 | 3.212e-2 | 6.005e-2 | 6.621e-2 | 6.646e-2 | 6.647e-2 | 6.647e-2 |
| | **22.32** | **32.33** | **13.51** | **14.77** | **14.82** | **14.82** | **14.82** |
| 256 | 4.771e-4 | 8.268e-3 | 3.073e-2 | 3.328e-2 | 3.354e-2 | 3.354e-2 | 3.354e-2 |
| | **22.59** | **32.77** | **14.20** | **14.89** | **15.00** | **15.00** | **15.00** |

We considered one a-priori-chosen layer-adapted mesh of Bakhvalov type:




**maximum nodal errors**

- The mesh is chosen so that the linear interpolation error $\|u - u^I\|_{\infty;\infty} \lesssim N^{-2}$.
- However, **as $\varepsilon \to 0$, the convergence rates deteriorate from $2$ to $1$**.
- E.g., for the final choice of $\varepsilon$ and $N$, the **aspect ratios** of the mesh elements take values **between 1 and 3.6e+8**.
- Considering these variations, the SECOND estimator performs reasonably well and its efficitivity indices stabilize as $\varepsilon \to 0$.
- By contrast, the FIRST estimator is adequate for $\varepsilon \sim 1$, but its effectivity deteriorates in the singularly perturbed regime.

Table: Bakhvalov mesh, $M = \frac{1}{2}N$: **energy-norm errors** and estimators.

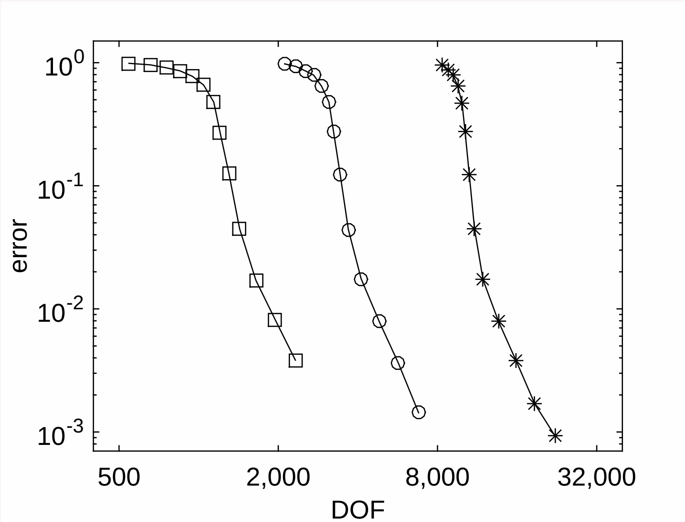
| $N$ | $\varepsilon = 1$ | $\varepsilon = 2^{-5}$ | $\varepsilon = 2^{-10}$ | $\varepsilon = 2^{-15}$ | $\varepsilon = 2^{-20}$ | $\varepsilon = 2^{-25}$ | $\varepsilon = 2^{-30}$ |
|---|---|---|---|---|---|---|---|
| | Errors (odd rows) & Computational Rates (even rows) | | | | | | |
| 64 | 3.202e-2 | 5.081e-3 | 7.993e-4 | 1.408e-4 | 2.489e-5 | 4.399e-6 | 7.777e-7 |
| | 1.00 | 0.99 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 128 | 1.602e-2 | 2.564e-3 | 3.991e-4 | 7.028e-5 | 1.242e-5 | 2.196e-6 | 3.882e-7 |
| | 1.00 | 0.99 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 256 | 8.011e-3 | 1.289e-3 | 1.997e-4 | 3.511e-5 | 6.207e-6 | 1.097e-6 | 1.940e-7 |
| | SECOND Estimator (odd rows) & **Effectivity Indices** (even rows) | | | | | | |
| 64 | 1.041e-1 | 2.102e-2 | 4.129e-3 | 7.393e-4 | 1.308e-4 | 2.311e-5 | 4.086e-6 |
| | **3.25** | **4.14** | **5.17** | **5.25** | **5.25** | **5.25** | **5.25** |
| 128 | 5.147e-2 | 1.051e-2 | 2.050e-3 | 3.711e-4 | 6.566e-5 | 1.161e-5 | 2.052e-6 |
| | **3.21** | **4.10** | **5.14** | **5.28** | **5.29** | **5.29** | **5.29** |
| 256 | 2.559e-2 | 5.269e-3 | 1.006e-3 | 1.858e-4 | 3.290e-5 | 5.817e-6 | 1.028e-6 |
| | **3.19** | **4.09** | **5.04** | **5.29** | **5.30** | **5.30** | **5.30** |

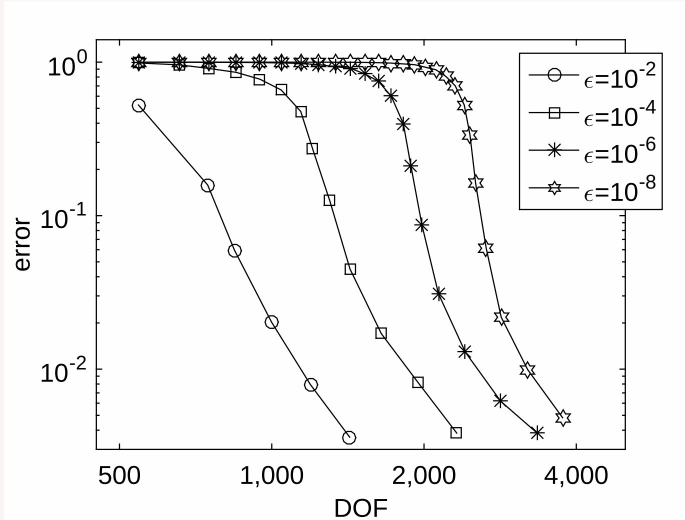
NOTE for $\varepsilon \ll 1$ : $\|u_h - u^I\|_{2\,;\Omega} \simeq \varepsilon\|\nabla u_h - (\nabla u)^I\|_{2\,;\Omega} \simeq \varepsilon^{1/2}N^{-1}$

$$\Rightarrow \boxed{\|u_h - u\|_{2\,;\Omega} \simeq |\!|\!|u_h - u|\!|\!|_{\varepsilon\,;\Omega} \simeq \varepsilon^{1/2}N^{-1} + N^{-2}}$$

Simple 2d TEST problem: $-\varepsilon^2 \Delta u + u = F(x)$ in $\Omega = (0,1)^2$ with $\varepsilon^2 = 10^{-6}$,
$u = 4y\,(1-y)\,[1 - x^2 - (e^{-x/\varepsilon} - e^{-1/\varepsilon})/(1 - e^{-x/\varepsilon})]$

Maximum errors for $\varepsilon = 10^{-4}$ and initial DOF varied (left), and $\varepsilon$ varied (right):




In each experiment, we started with a uniform mesh of right-angled triangles of diameter $H_T = 2^{-8}, 2^{-16}, 2^{-32}$, and aspect ratio $\frac{H_T}{h_T} = 2$. At each iteration, we marked for refinement the mesh elements responsible for at least 5% of the overall estimator $\mathcal{E}$, but no more than 15% of the elements. The marked elements were refined only in the $x$ direction using a single or triple green refinement (depending on the orientation of the mesh element). Edge swapping was also employed to improve geometric properties of the mesh and/or possibly reduce $\max_{T\in\mathcal{T}}\{\mathrm{osc}(f_h^I;T)\}$.

Why anisotropic meshes?

**Section A**

**Perceptions & expectations t.b. adjusted** for anisotropic meshes

**Section B**

Part 0 — Standard residual-type estimators on shape-regular meshes; their relation to interpolation errors

Part 1 — **Recent a posteriori estimates on anisotropic meshes**

Part 2 — **A bit of analysis**: 3 technical issues addressed...

Part 3 — **Some Numerics**

**Section C**

**Efficiency, i.e. lower estimator**: also problematic on anisotropic meshes...

**Lower Error Estimators on anisotropic meshes in the energy norm???**

(consistent with upper estimators?)

- **Standard Bubble Function Approach**

  This approach was employed by [Kunert & Verfürth 2000, Kunert 2001]: let $\varepsilon = 1$,

$$\underline{\mathcal{E}} := \Big\{ \sum_{S\in\mathcal{S}\backslash\partial\Omega} \varrho_S\, J_S^2 + \|h_T f_h^I\|_\Omega^2 \Big\}^{1/2} \lesssim \|u_h - u\|_{H^1(\Omega)} + \|h_T(f_h - f_h^I)\|_\Omega,$$

  For $S = \partial T_1 \cap \partial T_2$: $\boxed{\varrho_S = |S| \min\{h_{T_1},\, h_{T_2}\}}$

  We give **a numerical example** (for $\varepsilon = 1$) that clearly demonstrates that

  **short-edge jump residual terms in such bounds are not sharp**

**Lower Error Estimators on anisotropic meshes in the energy norm???**

(consistent with upper estimators?)

- **Standard Bubble Function Approach**

  This approach was employed by [Kunert & Verfürth 2000, Kunert 2001]: let $\varepsilon = 1$,

$$\underline{\mathcal{E}} := \Big\{ \sum_{S \in \mathcal{S} \setminus \partial\Omega} \varrho_S \, J_S^2 + \|h_T f_h^I\|_\Omega^2 \Big\}^{1/2} \lesssim \|u_h - u\|_{H^1(\Omega)} + \|h_T(f_h - f_h^I)\|_\Omega,$$

  For $S = \partial T_1 \cap \partial T_2$:

$$\boxed{\varrho_S = |S| \min\{h_{T_1}, h_{T_2}\}}$$

  We give **a numerical example** (for $\varepsilon = 1$) that clearly demonstrates that

  **short-edge jump residual terms in such bounds are not sharp**

- So, under additional restrictions on the anisotropic mesh, we shall give a **new bound for the short-edge jump residual terms**, and thus show that at least for some anisotropic meshes the error estimator constructed in the paper is efficient.

  For $\varepsilon = 1$ and $S = \partial T_1 \cap \partial T_2$:

$$\boxed{\varrho_S = |T_1 \cup T_2| = \text{local volume}}$$

Table 1: **Lower error estimators** for test problem with $u = \sin(\pi a x)$ and $\varepsilon = 1$.

| | $a=1$ | | | $a=3$ | | |
|---|---|---|---|---|---|---|
| | $N=20$ | $N=40$ | $N=80$ | $N=20$ | $N=40$ | $N=80$ |
| | Errors $\|u_h - u\|_{H^1(\Omega)}$ | | | | | |
| $M = 2N$ | 1.01e-1 | 5.04e-2 | 2.52e-2 | 9.26e-1 | 4.56e-1 | 2.27e-1 |
| $M = 8N$ | 1.01e-1 | 5.04e-2 | 2.52e-2 | 9.26e-1 | 4.56e-1 | 2.27e-1 |
| $M = 32N$ | 1.01e-1 | 5.04e-2 | 2.52e-2 | 9.26e-1 | 4.56e-1 | 2.27e-1 |
| $M = 128N$ | 1.01e-1 | 5.04e-2 | 2.52e-2 | 9.26e-1 | 4.56e-1 | 2.27e-1 |
| | $\underline{\mathcal{E}}$ with $\varrho_S = \|S\| \min\{h_{T_1}, h_{T_2}\}$ (odd rows) & **Effectivity Indices** (even rows) | | | | | |
| $M = 2N$ | 2.89e-1 | 1.45e-1 | 7.24e-2 | 2.51e+0 | 1.26e+0 | 6.33e-1 |
| | **2.87** | **2.88** | **2.88** | **2.72** | **2.78** | **2.79** |
| $M = 8N$ | 1.32e-1 | 6.59e-2 | 3.30e-2 | 1.17e+0 | 5.86e-1 | 2.93e-1 |
| | **1.31** | **1.31** | **1.31** | **1.26** | **1.29** | **1.29** |
| $M = 32N$ | 6.27e-2 | 3.14e-2 | 1.57e-2 | 5.62e-1 | 2.82e-1 | 1.41e-1 |
| | **0.62** | **0.62** | **0.62** | **0.61** | **0.62** | **0.62** |
| $M = 128N$ | 3.10e-2 | 1.55e-2 | 7.75e-3 | 2.79e-1 | 1.39e-1 | 6.97e-2 |
| | **0.31** | **0.31** | **0.31** | **0.30** | **0.31** | **0.31** |

**Standard Bubble Function Approach** $\Rightarrow$ **Lower Estimator NOT SHARP**

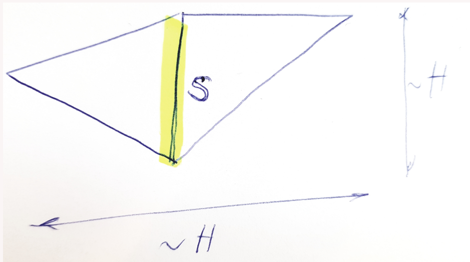
Table 2: **Lower error estimators** for test problem with $u = \sin(\pi a x)$ and $\varepsilon = 1$.

| | $a = 1$ | | | $a = 3$ | | |
|---|---|---|---|---|---|---|
| | $N = 20$ | $N = 40$ | $N = 80$ | $N = 20$ | $N = 40$ | $N = 80$ |
| | Errors $\|u_h - u\|_{H^1(\Omega)}$ | | | | | |
| $M = 2N$ | 1.01e-1 | 5.04e-2 | 2.52e-2 | 9.26e-1 | 4.56e-1 | 2.27e-1 |
| $M = 8N$ | 1.01e-1 | 5.04e-2 | 2.52e-2 | 9.26e-1 | 4.56e-1 | 2.27e-1 |
| $M = 32N$ | 1.01e-1 | 5.04e-2 | 2.52e-2 | 9.26e-1 | 4.56e-1 | 2.27e-1 |
| $M = 128N$ | 1.01e-1 | 5.04e-2 | 2.52e-2 | 9.26e-1 | 4.56e-1 | 2.27e-1 |
| | $\underline{\mathcal{E}}$ with $\varrho_S = \|T_1 \cup T_2\|$ (odd rows) & **Effectivity Indices** (even rows) | | | | | |
| $M = 2N$ | 3.00e-1 | 1.50e-1 | 7.52e-2 | 2.61e+0 | 1.32e+0 | 6.59e-1 |
| | **2.98** | **2.98** | **2.98** | **2.82** | **2.89** | **2.90** |
| $M = 8N$ | 2.51e-1 | 1.26e-1 | 6.28e-2 | 2.25e+0 | 1.13e+0 | 5.64e-1 |
| | **2.49** | **2.49** | **2.49** | **2.43** | **2.47** | **2.48** |
| $M = 32N$ | 2.47e-1 | 1.23e-1 | 6.18e-2 | 2.21e+0 | 1.11e+0 | 5.56e-1 |
| | **2.45** | **2.45** | **2.45** | **2.39** | **2.44** | **2.45** |
| $M = 128N$ | 2.46e-1 | 1.23e-1 | 6.17e-2 | 2.21e+0 | 1.11e+0 | 5.55e-1 |
| | **2.44** | **2.45** | **2.45** | **2.39** | **2.43** | **2.45** |

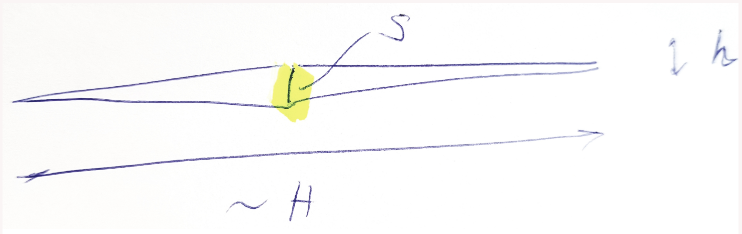
**New Lower Error Estimator** $\Rightarrow$ **EFFICIENT**

Where is the **issue with the standard bubble function approach for short-edge jump residual terms**?

- Essentially, the edge bubble works as a cut-off function
- Its gradient is $O(H^{-1})$ on shape-regular meshes




- For short edges on anis. meshes, the gradient of the edge bubble becomes $O(h^{-1})$

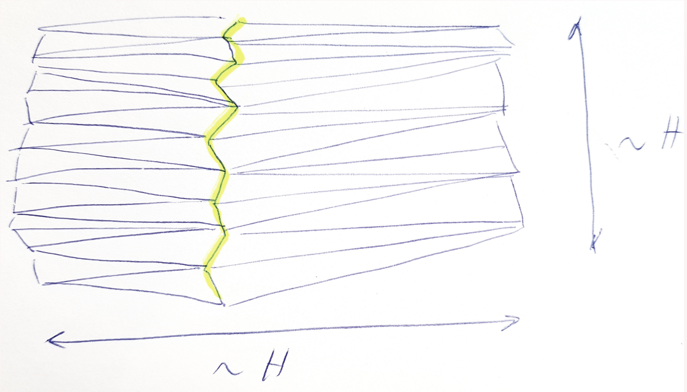



$\Rightarrow$ an ”incorrect” $H/h$ in the resulting estimator

- Note: no issue for long edges, as $|S|/(h^{-1}) \simeq hH \simeq$ local volume...

**How we rectify this?** (more detail)

- By looking at a **patch of anisotropic elements** of width $\simeq H$ and total area $\simeq H^2$

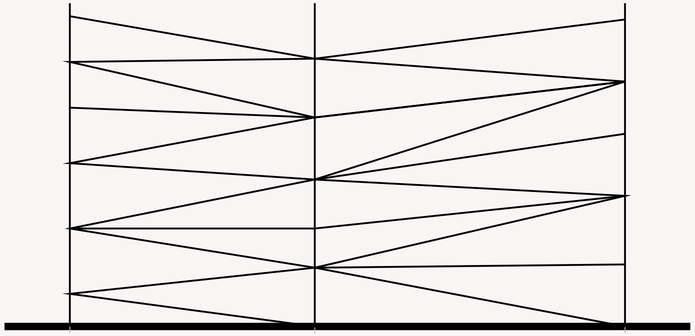



$\Rightarrow$ this allows us to use a cut-off function with a ”correct” gradient $O(H^{-1})$

- Unlike the single-edge-setting, the short-edge $J_S$ changes within the patch, so requires a more careful treatment...
- It’s not a full story... (also have to take care of the long edges within the patch...)
- Overall, as the setting is more complex, so the proof is more complex as well...

**How we rectify this?** (more detail)

- Consider a **partially structured mesh**:




**Theorem** [Short-edge jump residual terms]

$$\sum_{S\in\mathcal{S}\cap\{x=x_i\}} |\omega_S| J_S^2 \lesssim \|u_h - u\|^2_{H^1(\Omega_i)} + \|H_T \operatorname{osc}(f_h\,; T)\|^2_{\Omega_i}$$

Here $|\omega_S| \sim$ local volume [Kopteva, preprint, 2017, §9]

- **More general setting**: in preparation

(the proof is complete for both $\varepsilon = 1$ and $\varepsilon \ll 1$)

Why anisotropic meshes?

**Section A**

**Perceptions & expectations t.b. adjusted** for anisotropic meshes

**Section B**

Part 0 — Standard residual-type estimators on shape-regular meshes; their relation to interpolation errors

Part 1 — **Recent a posteriori estimates on anisotropic meshes**

Part 2 — **A bit of analysis**: 3 technical issues addressed...

Part 3 — **Some Numerics**

**Section C**

**Efficiency, i.e. lower estimator**: also problematic on anisotropic meshes...

- N. Kopteva, *Linear finite elements may be only first-order pointwise accurate on anisotropic triangulations*, *Math. Comp.*, 2014.

- A. Demlow and N. Kopteva, *Maximum-norm a posteriori error estimates for singularly perturbed elliptic reaction-diffusion problems*, Numer. Math., 2015.
- N. Kopteva, *Maximum-norm a posteriori error estimates for singularly perturbed reaction-diffusion problems on anisotropic meshes*, *SIAM J. Numer. Anal.*, 2015.
- N. Kopteva, *Energy-norm a posteriori error estimates for singularly perturbed reaction-diffusion problems on anisotropic meshes*, *Numer. Math.*, 2017

- N. Kopteva, *Fully computable a posteriori error estimator using anisotropic flux equilibration on anisotropic meshes*, 2017, submitted for publication, http://www.staff.ul.ie/natalia/pubs.html.

## Final

Thank you!

**$L_\infty$ norm**

Our <u>FIRST ESTIMATOR</u> reduces to

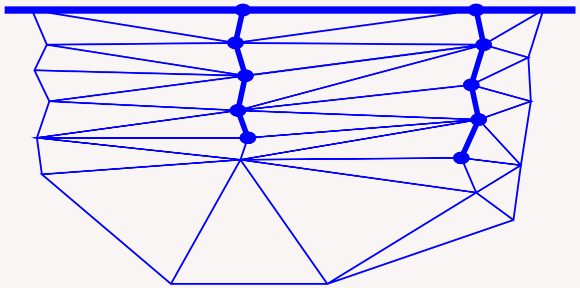
$$\|u_h - u\|_\infty \leqslant C\,\ell_h \max_{z\in\mathcal{N}}\Big(\min\{\varepsilon, H_z\}\,\big\|[\![\nabla u_h]\!]\big\|_{\infty\,;\gamma_z} + \min\{1, \tfrac{H_z^2}{\varepsilon^2}\}\,\|f_h^I\|_{\infty\,;\omega_z}\Big) + C\,\|f_h - f_h^I\|_{\infty\,;\Omega}\,,$$

**$C$ is independent of the diameters and the aspect ratios of elements in $\mathcal{T}$, and of $\varepsilon$.**

Here $f_h = f(\cdot, u_h)$, $\mathcal{N}$ is the set of nodes in $\mathcal{T}$, $[\![\nabla u_h]\!]$ is the standard jump in the normal derivative of $u_h$ across an element edge, $\omega_z$ is the patch of elements surrounding any $z \in \mathcal{N}$, $\gamma_z$ is the set of edges in the interior of $\omega_z$, $H_z = \text{diam}(\omega_z)$, $\ell_h = \ln(2 + \varepsilon\underline{h}^{-1})$, and $\underline{h}$ is the minimum height of triangles in $\mathcal{T}$.

- For $\varepsilon = 1$, this gives a standard a posteriori error bound, similar to [Eriksson, Nochetto, Nochetto et al], only now we prove it for anisotropic meshes.
- For $\varepsilon \in (0, 1]$, this is almost identical with our estimator for shape-regular case (on the previous page), but now we assume no shape regularity of the mesh.

**$L_\infty$ norm** In order to give a sharper (and more anisotropic in nature) bound for the interior-residual component of the error, we identify sequences of short edges that connect anisotropic nodes:



Under some additional assumptions on each such sequence (which we call a Path), our SECOND ESTIMATOR

$$\|u_h - u\|_\infty \leqslant C\,\ell_h\Big[\max_{z\in\mathcal{N}}\Big(\min\{\varepsilon, H_z\}\|J_z\|_{\infty\,;\gamma_z}\Big) + \max_{z\in\mathcal{N}\setminus\mathcal{N}_{\text{paths}}}\Big(\min\{1,\varepsilon^{-2}H_z^2\}\|f_h^I\|_{\infty\,;\omega_z}\Big)$$
$$+ \max_{z\in\mathcal{N}_{\text{paths}}}\Big(\min\{\varepsilon, H_z\}\min\{\varepsilon, h_z\}\|\varepsilon^{-2}f_h^I\|_{\infty\,;\omega_z} + \min\{1,\varepsilon^{-2}H_z^2\}\operatorname{osc}(f_h^I;\omega_z)\Big)\Big]$$
$$+ C\,\|f_h - f_h^I\|_{\infty\,;\Omega}\,,$$

**$C$ is independent of the diameters and the aspect ratios of elements in $\mathcal{T}$, and of $\varepsilon$.**

Here $\mathcal{N}_{\text{paths}}$ is the set of mesh nodes that appear in any path, $h_z \sim H_z^{-1}|\omega_z|$, $J_z = [\![\nabla u_h]\!]$.

Energy norm

our FIRST ESTIMATOR reduces to

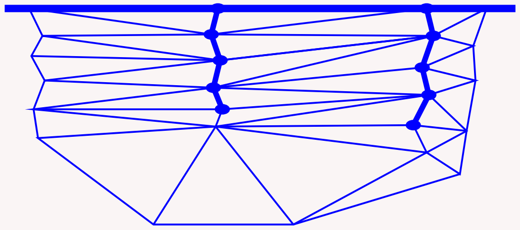
$$|||u_h - u|||_{\varepsilon\,;\Omega} \leqslant C\Big\{\sum_{z\in\mathcal{N}}\Big(\min\{1, \tfrac{\varepsilon}{h_z}\}\, h_z H_z \|\varepsilon[\![\nabla u_h]\!]\|^2_{\infty\,;\gamma_z} + \Big\|\min\Big\{1, \tfrac{H_z}{\varepsilon}\Big\}\, f_h^I\Big\|^2_{2\,;\omega_z}\Big)\Big\}^{1/2} + C\,\|f_h - f_h^I\|_{2\,;\Omega}\,,$$

**$C$ is independent of the diameters and the aspect ratios of elements in $\mathcal{T}$, and of $\varepsilon$.**

Here $f_h = f(\cdot, u_h)$, $\mathcal{N}$ is the set of nodes in $\mathcal{T}$, $[\![\nabla u_h]\!]$ is the standard jump in the normal derivative of $u_h$ across an element edge, $\omega_z$ is the patch of elements surrounding any $z \in \mathcal{N}$, $\gamma_z$ is the set of edges in the interior of $\omega_z$, $H_z = \mathrm{diam}(\omega_z)$, and $h_z \sim H_z^{-1}|\omega_z|$.

- For $\varepsilon = 1$, this gives a standard a posteriori error bound, similar to [Babuška et al], only now we prove it for anisotropic meshes.
- For $\varepsilon \in (0, 1]$, this is almost identical with our estimator for shape-regular case [Verfürth], but now we assume no shape regularity of the mesh.

**Energy norm** For a sharper ( bound for the interior-residual component of the error, we again identify sequences of short edges that connect anisotropic nodes:



Under some additional assumptions on each such sequence (which we call a Path), our SECOND ESTIMATOR

$$|||u_h - u|||_{\varepsilon\,;\Omega} \leqslant C\Big\{\sum_{z\in\mathcal{N}} \min\{1, \tfrac{\varepsilon H_z}{h_z^2}\}\, h_z H_z \|\varepsilon[\![\nabla u_h]\!]\|^2_{\infty\,;\gamma_z} + \sum_{z\in\mathcal{N}\setminus\mathcal{N}_{\text{paths}}} \|\min\{1, \tfrac{H_z}{\varepsilon}\}\, f_h^I\|^2_{2\,;\omega_z}$$

$$+ \sum_{z\in\mathcal{N}_{\text{paths}}} \Big(\|\min\{1, \tfrac{h_z}{\varepsilon}\}\, f_h^I\|^2_{2\,;\omega_z} + \|\min\{1, \tfrac{H_z}{\varepsilon}\}\, \text{osc}(f_h^I;\omega_z)\|^2_{2\,;\omega_z}\Big)\Big]$$

$$+ C\, \|f_h - f_h^I\|_{2\,;\Omega}\,,$$

**$C$ is independent of the diameters and the aspect ratios of elements in $\mathcal{T}$, and of $\varepsilon$.**

Here $\mathcal{N}_{\text{paths}}$ is the set of mesh nodes that appear in any path, $h_z \sim H_z^{-1}|\omega_z|$