Convexification methods for a 1-d coefficient inverse problem with experimental data

A.E. Kolesov

II International conference
Multiscale methods and Large-scale Scientific Computing

Moscow, August 16, 2018

Introduction

This work is one of the result of my work with M.V. Klibanov at UNC Charlotte, North Carolina, USA.

We develop a new version of the so-called *convexification* globally convergent numerical method for a 1-d coefficient inverse problem. We demonstrate its performance for both computationally simulated and experimental data.

Convexification is the method, which constructs globally strictly convex Tikhonov-like functionals for Coefficient Inverse Problems. The key element of such functionals is the presence of the Carleman Weight Function.

Convexification addresses the well known problem of multiple local minima and ravines of conventional Tikhonov-like functionals.

Outlook

Problem statement

Cost functional

Numerical implementation

Reconstruction results

Helmholtz equation

Consider the 1-d Helmholtz equation for the function $u(x,k)$,

$$u'' + k^2 c(x)\, u = -\delta(x - x_0), \quad x \in \mathbb{R},$$

$$\lim_{x\to\infty} (u' + iku) = 0, \lim_{x\to-\infty} (u' - iku) = 0.$$

where $c(x)$ is the spatially distributed dielectric constant, k is wave number.

Let $u_0(x,k)$ be the solution of the problem for the case $c(x) \equiv 1$. Then

$$u_0(x,k) = \frac{\exp(-ik|x - x_0|)}{2ik}.$$

Inverse problem

Let $\underline{k}$ and $\overline{k}$ be two positive constants and $\underline{k} < \overline{k}$. Our inverse problem is stated as:

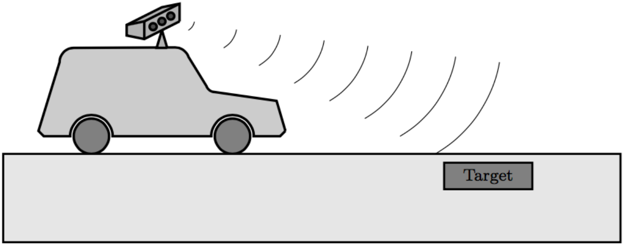
Coefficient Inverse Problem (CIP). *Determine the function $c(x)$, assuming that the following function $g_0(k)$ is given:*

$$g_0\left(k\right) = \frac{u(0,k)}{u_0(0,k)}, \quad k \in [\underline{k}, \overline{k}].$$

Existence and uniqueness of the solution $u\left(x,k\right)$ for each $k > 0$ was established in:

- M. V. Klibanov, L. H. Nguyen, A. Sullivan, and L. Nguyen, *A globally convergent numerical method for a 1-d inverse medium problem with experimental data*, Inverse Probl. Imaging, 10 (2016), pp. 1057–1085.

Forward Looking Radar



- L. Nguyen, D. Wong, M. Ressler, F. Koenig, B. Stanton, G. Smith, J. Sichina, and K. Kappra, *Obstacle avoidance and concealed target detection using the Army Research Lab ultra-wideband synchronous impulse reconstruction (UWB SIRE) forward imaging radar*, Proc. SPIE, 6553 65530H (2007), pp. 1–8.

Additional information

Denote

$$w\left(x,k\right)=\frac{u\left(x,k\right)}{u_0\left(x,k\right)}.$$

Then

$$w''+k^2\beta(x)w+2ikw'=0,\quad \beta(x)=c(x)-1.$$

$$w\left(0,k\right)=g_0\left(k\right),\quad k\in[\underline{k},\overline{k}].$$

Besides, it can be shown that

$$w'\left(0,k\right)=g_1\left(k\right)=2ik\left(g_0\left(k\right)-1\right),\quad k\in[\underline{k},\overline{k}].$$

$$w'(1,k)=0,\quad k\in[\underline{k},\overline{k}].$$

Asymptotic behavior

The following asymptotic behavior of the function $u(x,k)$ takes place:

$$u(x,k) = \frac{1}{2ikc^{1/4}(x)} \exp\left[-ik\int\limits_{x_0}^{x}\sqrt{c(\xi)}d\xi\right](1+\widehat{u}(x,k)), \qquad k\to\infty, \quad \forall x\in[0,1],$$

$$\widehat{u}(x,k) = O\left(\frac{1}{k}\right), \quad \partial_k\widehat{u}(x,k) = O\left(\frac{1}{k^2}\right), \quad k\to\infty.$$

- M. V. Klibanov, L. H. Nguyen, A. Sullivan, and L. Nguyen, *A globally convergent numerical method for a 1-d inverse medium problem with experimental data*, Inverse Probl. Imaging, 10, pp. 1057–1085, 2016.

Differential equation

Using the asymptotic expansion, we can prove that there exists unique function $v(x,k)$, $k>0$ such that

$$v(x,k) = \frac{\log w(x,k)}{k^2}$$

Then our equation becomes:

$$v'' + k^2 \left(v'\right)^2 - 2ikv' = -\beta(x), \quad k \in [\underline{k}, \overline{k}],$$

$$v(0,k) = \frac{\log g_0(k)}{k^2}, \quad v'(0,k) = \frac{2i}{k}\left(1 - \frac{1}{g_0(k)}\right), \quad v'(1) = 0$$

Differentiate this equation with respect to k

$$v_k'' + 2k^2 v_k' v' + 2k \left(v'\right)^2 - 2ikv_k' - 2iv' = 0.$$

Truncated Fourier-like series

We assume that the function $v(x,k)$ can be represented via a truncated Fourier series

$$v(x,k) = \sum_{n=0}^{N-1} y_n(x)\, \psi_n(k),$$

where $\{\psi_n(k)\}_{n=0}^{\infty}$ is an orthonormal basis of real valued function such that the following two conditions are met:

1. The first derivative with respect to k of any element of this basis is not identically zero,
2. This derivative should be a linear combination of a finite number of elements of this basis.

- M. V. Klibanov. *Convexification of restricted dirichlet-to-neumann map.* J. Inverse and Ill-Posed Problems, 25: pp. 669 – 685, 2017.

Special orthonormal basis

Consider the set of functions

$$\{\kappa^n e^{\kappa}\}_{n=0}^{\infty}, \quad \kappa = \frac{k-\underline{k}}{\overline{k}-\underline{k}} \in [\underline{k}, \overline{k}].$$

We orthonormalize it using the classical Gram-Schmidt orthonormalization procedure and obtain the orthonormal basis

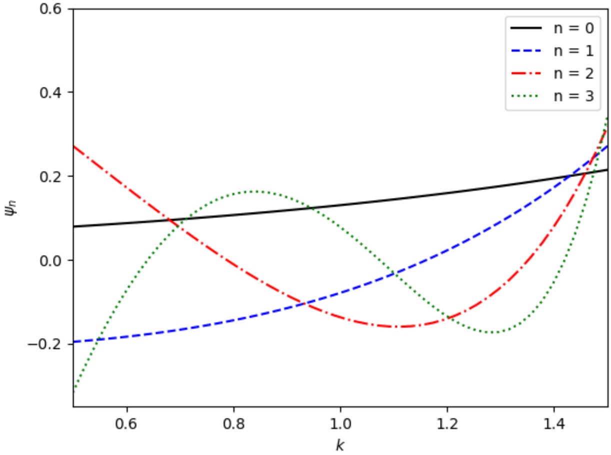
$$\{\psi_n(k)\}_{n=0}^{\infty}$$

Each function $\psi_n(k)$ has the form

$$\psi_n(k) = p_n(k)\, e^{k},$$

where $p_n(k)$ is the polynomial of the degree n.

Special orthonormal basis



Invertible matrix

$$\psi_n'(k) = p_n(k)\,e^k + p_n'(k)\,e^k = \psi_n(k) + \sum_{j=0}^{n-1} b_{jn}\psi_j(k)\,.$$

It can be proven that

$$a_{mn} = (\psi_n', \psi_m) = \begin{cases} 1 \text{ if } n = m, \\ 0 \text{ if } n < m. \end{cases}$$

Then $A = (a_{mn})_{m,n=0}^{N-1}$ is an upper triangular matrix:

$$A = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with $\det(A) = 1$. Thus, the inverse matrix A^{-1} exists.

Coupled quasilinear equations

We obtain

$$\sum_{n=0}^{N-1} y_n''(x)\,\psi_n'(k) + 2k^2 \sum_{n,m=0}^{N-1} y_n'(x)\,y_m'(x)\,\psi_n'(k)\,\psi_m'(k)$$

$$+2k\left[\sum_{n=0}^{N-1} y_n'(x)\,\psi_n(k)\right]^2 - 2ik\sum_{n=0}^{N-1} y_n'(x)\,\psi_n'(k)$$

$$-2i\sum_{n=0}^{N-1} y_n'(x)\,\psi_n(k) = 0.$$

Multiply this equation by $\psi_s(k)\,,\, s\in[0,N-1]$ and integrate with respect to $k\in\left(\underline{k},\overline{k}\right)$.

Coupled quasilinear equations

We obtain

$$Ay'' + \widetilde{F}\left(y'\right) = 0.$$

where $y\left(x\right) = \left(y_0, ..., y_{N-1}\right)^T\left(x\right)$ is $N-$D vector function. Since the matrix A is invertible, we multiply this equation by A^{-1} and obtain

$$L(y) = y'' + F\left(y'\right) = 0, \quad x \in \left[0, 1\right],$$

$$y\left(0\right) = f_0, \quad y'\left(0\right) = f_1, \quad y'\left(1\right) = 0.$$

where the vector function $F\left(y'\right) = M_N^{-1}\widetilde{F}\left(y'\right).$

Carleman weight function

$$\varphi_\lambda\left(x\right) = e^{-\lambda x}.$$

Lemma (Carleman estimate). *For any complex valued function $u \in H^2\left(0,1\right)$ with $u(0) = u'(0) = 0$ and for any parameter $\lambda > 1$ the following Carleman estimate holds*

$$\int\limits_0^1 \left|u''\right|^2 \varphi_\lambda^2 dx \geq C \int\limits_0^1 |u''|^2 \varphi_\lambda^2 dx +$$

$$+ C\lambda \int\limits_0^1 |u'|^2 \varphi_\lambda^2 dx + C\lambda^3 \int\limits_0^1 |u|^2 \varphi_\lambda^2 dx,$$

where the constant $C > 0$ is independent of u and λ.

Zero boundary conditions

We have to arrange zero Dirichlet and Neumann boundary conditions at $x = 0, 1$ for a new vector function p, which is associated with the vector function y as:

$$p(x) = y(x) - f(x), \quad p(0) = p'(0) = p'(1) = 0,$$

where $f(x)$ is a certain function with $f(0) = f_0$, $f'(0) = f_1$, $f'(1) = 0$.

We are doing so because some of our theorems are applicable only in the case of zero boundary conditions.

- A.B. Bakushinskii, M.V.Klibanov and N.A. Koshev, *Carleman weight functions for a globally convergent numerical method for ill-posed Cauchy problems for some quasilinear PDEs*, Nonlinear Analysis: Real World Applications, 34 (2017), pp. 201–224.

Minimization Problem

Let $R > 0$ be an arbitrary number. Consider the set $B(R)$ of functions $p(x)$ defined as:

$$B(R) = \left\{ p \in H^2(0,1) : p(0) = p'(0) = p'(1) = 0, \|p\|_{H^2(0,1)} < R \right\}.$$

Our Tikhonov-like weighted functional is

$$J_{\lambda,\alpha}(p) = e^{2\lambda} \int_0^1 |L(p)|^2 \varphi_\lambda^2 dx + \alpha \|p\|_{H^2(0,1)}^2 .$$

where $\alpha \in (0,1)$ is the regularization parameter.

Minimization Problem. *Minimize* $J_{\lambda,\alpha}(p)$ *on the set* $p \in \overline{B(R)}$.

Theorems

Theorem 1. (strict convexity) *The functional* $J_{\lambda,\alpha}(p)$ *has the Frechét derivative* $J'_{\lambda,\alpha}(p)$ *at each point* $p \in B(2R)$. *Also, there exists a number* $\lambda_1 = \lambda_1(R, F, N) > 1$ *such that for all* $\lambda \geq \lambda_1$ *the functional* $J_{\lambda,\alpha}(p)$ *is strictly convex on the set* $\overline{B(R)}$, *i.e. for all* $p_1, p_2 \in \overline{B(R)}$:

$$J_{\lambda,\alpha}(p_2) - J_{\lambda,\alpha}(p_1) - J'_{\lambda,\alpha}(p_1)(p_2 - p_1) \geq C_1 \|p_2 - p_1\|^2_{H^2(0,1)}.$$

Theorem 2. *For any* $\lambda \geq \lambda_1$ *and for any* $\alpha \in (0, 1)$ *there exists a unique minimizer* $p_{\min,\lambda,\alpha}$ *of the functional* $J_{\lambda,\alpha}(p)$ *on the set* $\overline{B(R)}$. *Furthermore,*

$$J'_{\lambda,\alpha}(p_{\min,\lambda,\alpha})(p_{\min,\lambda,\alpha} - p) \leq 0, \quad \forall p \in \overline{B(R)}.$$

Algorithm

- Calculate the boundary conditions q_0, q_1 and then f_0, f_1
- Define the initial guess y_0 for the vector function y as $y_0 = f$.
- Minimize the functional $J_{\lambda,\alpha}(y)$. Then transform the found vector function y in the vector function v.
- Compute the approximation β_{comp}.

$$\beta_{comp} = -v'' - k_0^2 \left(v'\right)^2 + 2ik_0 v'$$

- After averaging $\beta_{comp}\left(x\right)$, determine the coefficient c_{comp} as follows:

$$c_{comp} = \begin{cases} \mathrm{Re}(\beta_{comp}) + 1.0, & \text{if } \mathrm{Re}(\beta_{comp}) \geq \rho \max(\mathrm{Re}(\beta_{comp})), \\ 1.0, & \text{otherwise} \end{cases}$$

where $\rho \in (0, 1)$ is the truncation factor.

Finite difference discretization

We divide the intervals $k \in [\underline{k}, \overline{k}]$ and $x \in [0, 1]$ into N_k and N_x equal subintervals, respectively, and obtain the $N_k \times N_x$ two dimensional mesh with the grid points (k_m, x_j): $k_m = \underline{k} + m h_k$, $m = 0, \ldots, N_k$, $x_j = j h_x$, $j = 0, \ldots, N_x$, $h_k = (\overline{k} - \underline{k})/N_k$, $h_x = 1.0/N_x$.

We need to find the discrete function $v = \{v_{m,j}\}$, where $v_{m,j} = v(k_m, x_j)$. The discrete version of Fourier-series can be written as

$$v = \psi y,$$

where $\psi = \{\psi_{m,n}\}$ is the $N_k \times N$ matrix with $\psi_{m,n} = \psi_n(k_m)$ and $y = \{y_{n,j}\}$ is the $N \times N_x$ two dimensional discrete vector function.

Discrete functional and its gradient

Consider

$$y_{n,j} = a_{n,j} + ib_{n,j}, \quad L_{n,j} = I_{n,j} + iS_{n,j}$$

Discrete functional:

$$J = e^{2\lambda} h_x \sum_{n=0}^{N-1} \sum_{j=0}^{N_x-1} |L_{n,j}|^2 \varphi_{\lambda_j}^2 = e^{2\lambda} h_x \sum_{n=0}^{N-1} \sum_{j=0}^{N_x-1} \left(I_{n,j}^2 + S_{n,j}^2\right) \varphi_{\lambda_j}^2$$

$$L_{n,j} = \frac{y_{n,j+1} - 2y_{n,j} + y_{n,j-1}}{h_x^2} - F\left(\frac{y_{n,j+1} - y_{n,j-1}}{2h_x}\right),$$

$$\varphi_{\lambda_j} = e^{\lambda h_x j}.$$

Discrete functional and its gradient

Discrete gradient:

$$\frac{\partial J}{\partial y_{s,l}} = \frac{1}{2}\left(\frac{\partial J}{\partial a_{s,l}} + i\frac{\partial J}{\partial b_{s,l}}\right), \; s = 0, \dots, N-1, \; l = 0, \dots, N_x - 1,$$

where

$$\frac{\partial J}{\partial a_{s,l}} = 2e^{2\lambda} h_x \sum_{n=0}^{N-1} \sum_{j=0}^{N_x-1} \left[I_{n,j} \frac{\partial I_{n,j}}{\partial a_{s,l}} + S_{n,j} \frac{\partial S_{n,j}}{\partial a_{s,l}} \right] \varphi_{\lambda_j}^2,$$

$$\frac{\partial J}{\partial b_{s,l}} = 2e^{2\lambda} h_x \sum_{n=0}^{N-1} \sum_{j=0}^{N_x-1} \left[I_{n,j} \frac{\partial I_{n,j}}{\partial b_{s,l}} + S_{n,j} \frac{\partial S_{n,j}}{\partial b_{s,l}} \right] \varphi_{\lambda_j}^2,$$

- L. Sorber, M. Van Barel, L. De Lathauwer *Unconstrained Optimization of Real Functions in Complex Variables*, SIAM J. Optim., 22(3), pp. 879–898. (2012)

Discrete functional and its gradient

It can be shown that $\dfrac{\partial I_{n,j}}{\partial a_{s,l}} = \dfrac{\partial S_{n,j}}{\partial b_{s,l}}, \quad \dfrac{\partial I_{n,j}}{\partial b_{s,l}} = -\dfrac{\partial S_{n,j}}{\partial a_{s,l}}.$

Then

$$\frac{\partial J}{\partial y_{s,l}} = e^{2\lambda} h_x \sum_{n=0}^{N-1} \sum_{j=0}^{N_x-1} \left[(I_{n,j} + iS_{n,j}) \frac{\partial I_{n,j}}{\partial a_{s,l}} + (S_{n,j} - iI_{n,j}) \frac{\partial I_{n,j}}{\partial b_{s,l}} \right] \varphi{\lambda}_j^2$$

where

$$\frac{\partial I_{n,j}}{\partial a_{s,l}} = \frac{\partial a''_{n,j}}{\partial a_{s,l}} + F_1 \left(\frac{\partial a'_{n,j}}{\partial a_{s,l}} \right), \quad \frac{\partial I_{n,j}}{\partial b_{s,l}} = F_2 \left(\frac{\partial a'_{n,j}}{\partial a_{s,l}} \right),$$

$$\frac{\partial a''_{n,j}}{\partial a_{s,l}} = \frac{\delta^{s,l}_{n,j+1} - 2\delta^{s,l}_{n,j} + \delta^{s,l}_{n,j-1}}{h_x^2}, \quad \frac{\partial a'_{n,j}}{\partial a_{s,l}} = \frac{\delta^{s,l}_{n,j+1} - \delta^{s,l}_{n,j-1}}{2h_x},$$

$$\delta^{s,l}_{n,j} = \begin{cases} 1, & n = s, i = j, \\ 0, & \text{otherwise} \end{cases}$$

Data generation

We solve 1d Lippmann-Schwinger (LS) equation

$$u(x,k) = \frac{e^{-ik|x-x_0|}}{2ik} + \frac{k}{2i}\int\limits_0^1 e^{-ik|x-\xi|}\left(c(\xi)-1\right)u(\xi,k)\,d\xi,$$

where the function $c(x)$ is set as follows:

$$c(x) := c_{true}(x) = \begin{cases} \widehat{c}_{true}, & x \in (x_{loc}-d/2, x_{loc}+d/2), \\ 1, & \text{elsewhere.} \end{cases}$$

Here, $\widehat{c}_{true}$ is the dielectric constant, x_{loc} is the location of the center, and d is width.

In our numerical experiments:

$$\widehat{c}_{true} = \{3.0,\ 4.0,\ 5.0,\ 6.0\}, \quad x_{loc} = \{0.1,\ 0.2,\ 0.3,\ 0.4\}, \quad d = 0.1$$

Data generation

We consider the interval of wave numbers

$$k_m \in [0.5,\, 1.5], \quad N_k = 10.$$

By solving LS equation for every point k_m we obtain the noiseless boundary function $g_0(k_m)$.
Next, we add the random noise

$$g_{0,\delta}(k_m) = g_0(k_m)(1.0 + \delta\sigma(k_m)),\, \sigma = \sigma_r(k_m) + i\sigma_r(k_m),$$

where δ is the noise level, σ_r and σ_i are random numbers, uniformly distributed between -1.0 and 1.0.

In our computations $\delta = 0.05$.

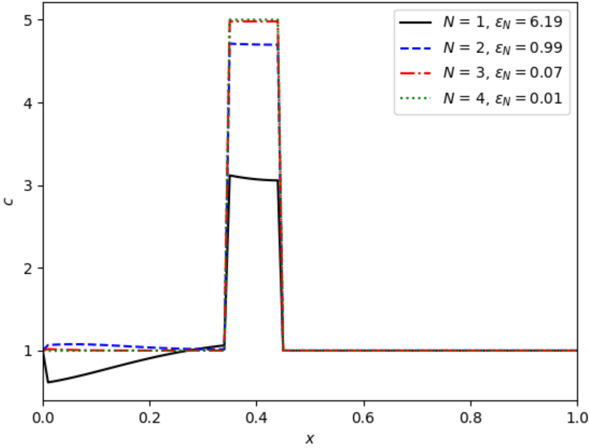
To reduce the noise the function $g_{0,\delta}(k_n)$ is smoothed out by using the standard averaging procedure

Optimal number of basis functions

We need to determine the optimal number N of terms in the truncated Fourier series:

- We solve the LS equation for a reference target with $c(x) := \widehat{c}_{true}(x)$.
- We obtain the functions $w_{true}(x, k)$ and $v_{true}(x, k)$.
- We compute vector functions $y_{true,N}(x)$ for different values of N
- Reconstruct approximate functions $c_{appr,N}(x)$.

The approximate functions $c_{appr,N}(x)$



Optimal number of basis functions

We can see that the functions $c_{appr,N}(x)$ are accurately approximated for both $N = 3$ and $N = 4$, and their approximation errors

$$\varepsilon_N = \|c_{appr,N} - c_{true}\|_{L^2(0,1)}$$

are sufficiently small: $\varepsilon_N = 0.07$ and 0.01, respectively.

Therefore, we choose the optimal number of functions in our basis $N = 3$.

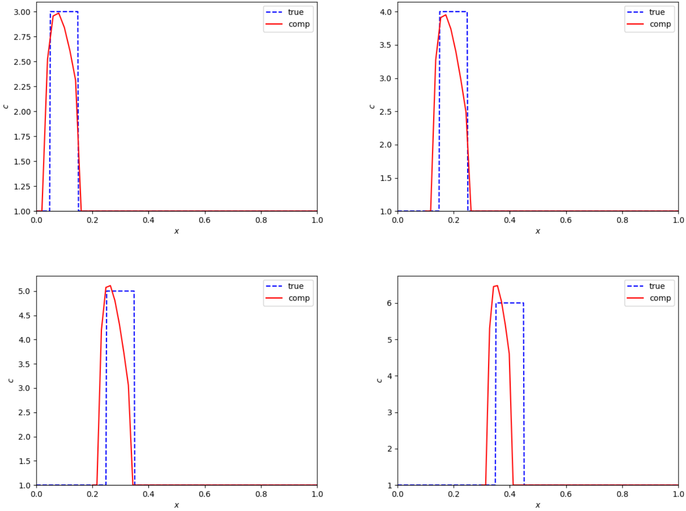
Reconstruction results for simulated targets.

$\widehat{c}_{true}$	x_{loc}	$\widehat{c}_{comp}$	ε_{comp}, %	$\widehat{c}_{true}$	x_{loc}	$\widehat{c}_{comp}$	ε_{comp}, %
3.0	0.1	2.98	0.67	5.0	0.1	5.32	6.40
	0.2	3.13	4.33		0.2	5.14	2.80
	0.3	2.80	6.67		0.3	5.11	2.20
	0.4	3.17	5.67		0.4	5.19	3.80
4.0	0.1	4.28	7.00	6.0	0.1	6.19	3.17
	0.2	3.95	1.25		0.2	6.25	4.17
	0.3	4.03	0.75		0.3	6.39	6.50
	0.4	4.12	3.00		0.4	6.47	7.83

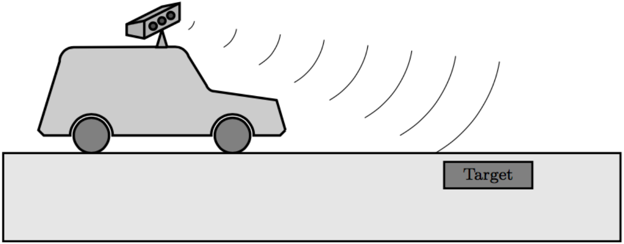
$$\widehat{c}_{comp} = \max(c_{comp}), \quad \varepsilon_{comp} = \frac{|\widehat{c}_{comp} - \widehat{c}_{true}|}{\widehat{c}_{true}} \cdot 100\%.$$

Parameters: $N_k = 10$, $N_x = 50$, $N = 3$, $\lambda = 3.0$, $\alpha = 0.05$.

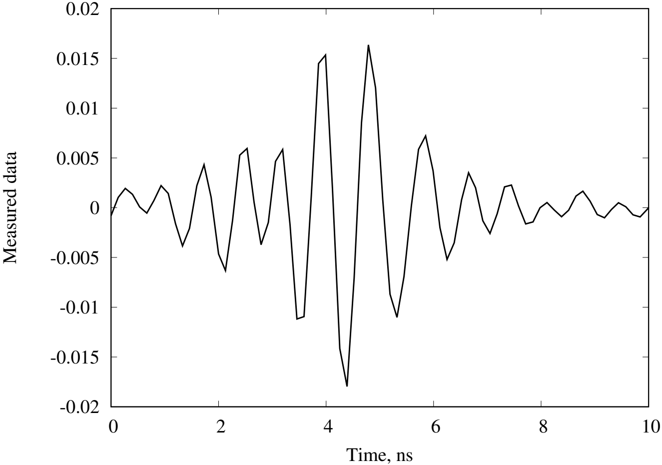
Reconstruction results for simulated targets.



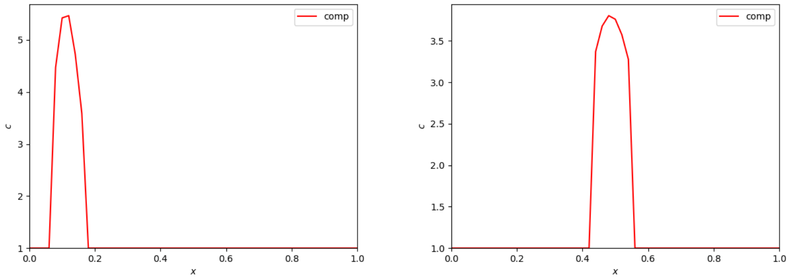
Schematic diagram of data collection



Measured time dependent data



Reconstruction results for bush (left) and wood (right)



Target	c_{comp}	c_{true}
Bush	5.47	$[3, 20]$
Wood	3.80	$[2, 6]$

Thank you for attention!