Mathematical modelling of the fluid flow and geo-mechanics in the fractured heterogeneous porous media using multiscale model reduction

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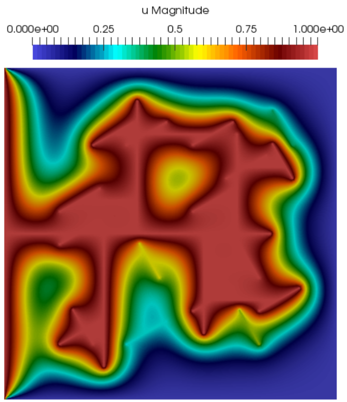
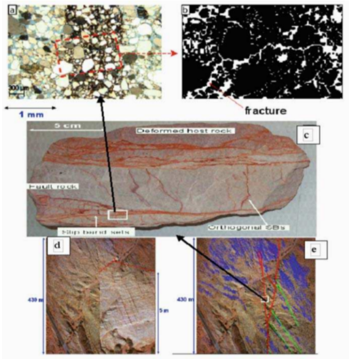
Introduction

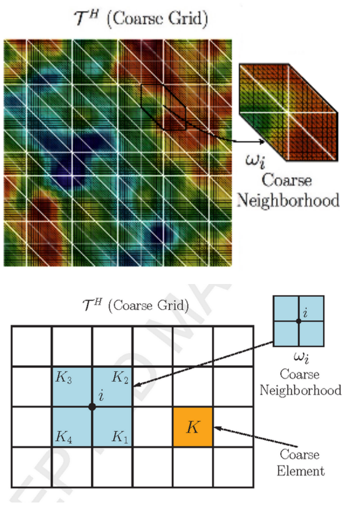
- Mathematical simulation of the flow processes in fractured porous media plays an important role in reservoir sumulation, nuclear waste disposal, unconventional gas production and geothermal energy production.
- Due to high permeability, fractures have a significant impact on the flow processes.
- Fractures have very small thickness compared to typical reservoir sizes. A common approach to model fracture media is to consider the fractures as lower-dimensional objects.
- Fracture networks have complex geometries and fine grid simulation of the processes in fractured porous media can be computationally expensive.





Motivation. Multiscale methods.

- To reduce the dimension of the fine scale system, multiscale methods or upscaling techniques are used.
- Multiscale methods should combine the simplicity and efficiency of a coarse-scale models and the accuracy of microscale approximations.
- To present the microscale interaction between the fractures and the matrix, we use Generalized Multiscale Finite Element Method (GMsFEM)



Problem formulation

Let $\Omega \subset R^d$ is computation domain. We consider a mathematical model of coupled flow and mechanics in fractured poroelastic medium. The balance of a linear momentum in the solid is given as

$$-\operatorname{div} \sigma_T(u, p) = 0, \quad \sigma_T(u, p) = \sigma(u) - \alpha p \mathcal{I}, \quad x \in \Omega,$$

Relation between the stress σ and strain ε tensors is given as

$$\sigma(u) = \lambda \varepsilon_v \mathcal{I} + 2\mu\varepsilon(u), \quad \varepsilon(u) = 0.5(\nabla u + (\nabla u)^T),$$

The fluid mass conservation is given as follows:

$$\frac{\partial m}{\partial t} + \operatorname{div}(\rho q) = \rho f, \quad q = -\frac{k}{\nu_f} \operatorname{grad} p, \quad x \in \Omega,$$

where m is the fluid mass, q is Darcy velocity, ν_f is the viscosity, ρ is the fluid density, and f is the source term. Here, for simplicity, we neglect the gravitational forces.

Fine grid approximation

For numerical solution of the poroelasticity problem on fine grid use a standard finite element method.
Find $(u, p) \in V \times Q$ such that

$$a_u(u, v) + b(p, v) = 0, \forall\ v \in \hat{V},$$

$$d\left(\frac{du}{dt}, q\right) + s\left(\frac{dp}{dt}, q\right) + a_p(p, q) = l(q), \quad \forall\ q \in \hat{Q}.$$

where $V = [H^1(\Omega)]^d$ and $Q = H^1(\Omega)$.

$$a_u(u, v) = \int_\Omega \sigma(u)\, \epsilon(v) dx, \quad a_p(p, q) = \int_\Omega \left(\frac{k}{\nu} \operatorname{grad} p, \operatorname{grad} q\right) dx,$$

$$c(p, q) = \int_\Omega \frac{1}{M}\, p\, q\, dx, \quad l(q) = \int_\Omega f\, q\, dx,$$

$$b(p, v) = \int_\Omega \alpha(\operatorname{grad} p, v) dx, \quad d(u, q) = \int_\Omega \alpha \operatorname{div} u\, q\, dx.$$

Fine grid approximation

The standard implicit finite difference scheme is used for the time approximation of the pressure equation and we solve following coupled system in the matrix form on the fine grid.

$$\frac{1}{\tau}\begin{pmatrix} M & D \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p-\hat{p} \\ u-\hat{u} \end{pmatrix} + \begin{pmatrix} A & 0 \\ B & K \end{pmatrix}\begin{pmatrix} p \\ u \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix},$$

where $M = [m_{ij}]$, $A = [a_{ij}]$, $K = [k_{ij}]$, $D = [d_{ij}]$, $B = [b_{ij}]$, $F = [f_j]$

$$m_{ij} = \int_\Omega c_m \phi_i \, \phi_j \, dx, \quad a_{ij} = \int_\Omega a_m \operatorname{grad} \phi_i \cdot \operatorname{grad} \phi_j \, dx,$$

$$k_{ij} = \int_\Omega \sigma(\Phi_i) : \epsilon(\Phi_j) \, dx, \quad d_{ij} = \int_\Omega \alpha \operatorname{grad} \phi_i \, \Phi_j \, dx,$$

$$b_{ij} = \int_\Omega \alpha \operatorname{div} \Phi_i \, \phi_j \, dx, \quad f_j = \int_\Omega f \, \phi_j \, dx,$$

and Φ_i and ϕ_i are the linear basis functions for displacements and pressure.

Coarse grid approximation using GMsFEM

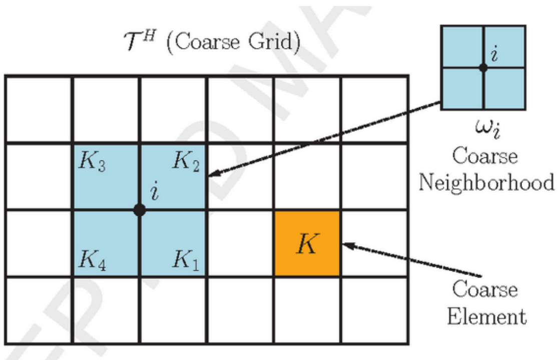
For coarse grid approximation of the poroelasticity problems in heterogeneous media, we use the Generalized Multiscale Finite Element Method (GMsFEM). GMsFEM contains following steps:

(1) Construction of the coarse and fine meshes,

(2) Generation of the local domains where we construct multiscale basis functions,

(3) Solution of the local spectral problems for multiscale basis functions construction,

(4) Construction and solution of the coarse scale approximation on multiscale space.



GMsFEM. Spectral problem

For construction of the multiscale basis functions, we solve a local spectral problems in domain ω_i for displacement and pressure separately.

- Pressure

$$a_p(\phi, q) = \lambda_p s_p(\phi, q),$$

$$a_p(\phi, q) = \int_{\omega_i} a_m \nabla\phi, \nabla q dx + \sum_l \int_{\gamma l} a_f \nabla\phi, \nabla q dx, \quad s_p(\phi, q) = \int_{\omega_i} k\, \phi\, q\, dx,$$

or

$$A_p\phi = \lambda_p S_p \phi.$$

- Displacement

$$a_u(\Phi, v) = \lambda_u s_u(\Phi, v),$$

$$a_u(\Phi, v) = \int_{\omega_i} \sigma(\Phi), \varepsilon(v) dx, \quad s_u(\Phi, v) = \int_{\omega_i} (\lambda + 2\mu)(\Phi, v) dx,$$

or

$$A_u\Phi = \lambda_u S_u \Phi.$$

We form the multiscale spaces V_h, Q_h using eigenvectors ϕ_1, ϕ_2, ..., ϕ_L, Φ_1, Φ_2,...,Φ_L corresponding to the first smallest L eigenvalues, where $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_L$.

GMsFEM. Snapshot space

For solution of the local spectral problem, we use a snapshot space V_{snap} for displacement and Q_{snap} for pressure.

- Pressure

$$a_p(\phi_j, q) = 0, \quad x \in \omega_i$$

$$\phi_j = \delta_j, \quad x \in \partial\omega_i.$$

- Displacement

$$a_u(\Phi_j, v) = 0, \quad x \in \omega_i$$

$$\Phi_j = (\delta_j, 0) \text{ or } (0, \delta_j), \quad x \in \partial\omega_i$$

We define a transition matrices R^u_{snap} and R^p_{snap}:

$$R^u_{snap} = [\Phi_1^{snap}, ..., \Phi_{L_i}^{snap}] \quad R^p_{snap} = [\phi_1^{snap}, ..., \phi_{L_i}^{snap}].$$

GMsFEM. Spectral problem

We define matrices for displacements and for pressure and solve following eigenvalue problem on the snapshot space.

- Displacement:
$$\overline{K}_\omega \overline{\Phi} = \lambda_u \overline{Q}_\omega \overline{\Phi},$$
$$\overline{K}_\omega = R^u_{snap} K_\omega (R^u_{snap})^T, \quad \overline{Q}_\omega = R^u_{snap} Q_\omega (R^u_{snap})^T,$$
where $\Phi^\omega_j = (R^u_{snap})^T \overline{\Phi}_j$.
- Pressure:
$$\overline{A}_\omega \overline{\phi} = \lambda_p \overline{S}_\omega \overline{\phi},$$
$$\overline{A}_\omega = R^p_{snap} A_\omega (R^p_{snap})^T, \quad \overline{S}_\omega = R^p_{snap} S_\omega (R^p_{snap})^T,$$
where $\phi^\omega_j = (R^p_{snap})^T \overline{\phi}_j$.

Coarse grid approximation using GMsFEM

For obtaining conforming basis functions we use linear partition of unity functions. We construct transition matrices R_u and R_p from a fine grid to a coarse grid and use it for reducing the dimension of the problem.

$$R = \begin{pmatrix} R_u & 0 \\ 0 & R_p \end{pmatrix}$$

with

$$R_u = \{\chi^1\Phi_1^1, \chi^1\Phi_2^1, ..., \chi^1\Phi_L^1, ..., \chi^{N_c}\Phi_1^{N_c}, \chi^{N_c}\Phi_2^{N_c}, ..., \chi^{N_c}\Phi_L^{N_c}\}$$

and

$$R_p = \{\chi^1\phi_1^1, \chi^1\phi_2^1, ..., \chi^1\phi_L^1, ...\chi^{N_c}\phi_1^{N_c}, \chi^{N_c}\phi_2^{N_c}, ..., \chi^{N_c}\phi_L^{N_c}, \}.$$

where χ^i is linear partition of unity functions, L is the number of basis functions and N_c is the number of vertices of a coarse grid.

Coarse grid approximation using GMsFEM

Then the system of equations can be translated into a coarse grid

$$\frac{1}{\tau}\begin{pmatrix} M_c & D_c \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p_c - \check{p}_c \\ u_c - \check{u}_c \end{pmatrix} + \begin{pmatrix} A_c & 0 \\ B_c & K_c \end{pmatrix}\begin{pmatrix} p_c \\ u_c \end{pmatrix} = \begin{pmatrix} F_c \\ 0 \end{pmatrix},$$

where

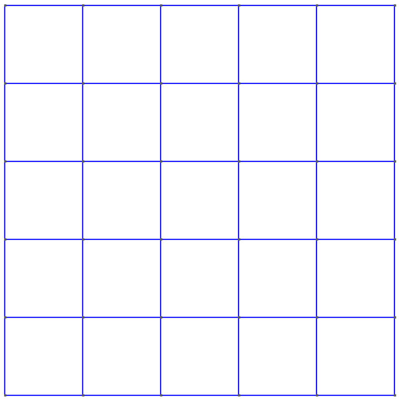
$$K_c = R_u K R_u^T, \quad A_c = R_p A R_p^T, \quad M_c = R_p M R_p^T,$$

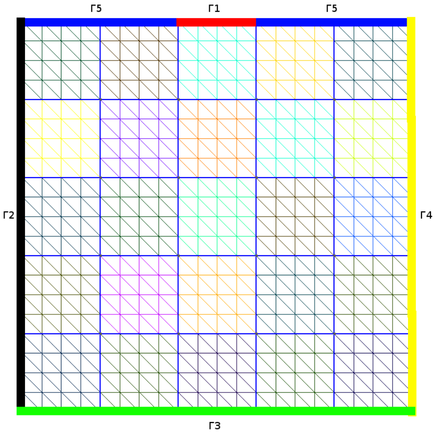
$$B_c = R_u B R_p^T, \quad D_c = R_p D R_u^T, \quad F_c = R_p F.$$

After obtaining of a coarse-scale solution, we can reconstruct fine-scale solution

$$u_{ms} = R_u^T u_c, \quad p_{ms} = R_p^T p_c.$$

Numerical results for 2D





Computational meshes. Left: coarse grid, Right: fine grid.

Boundary conditions:

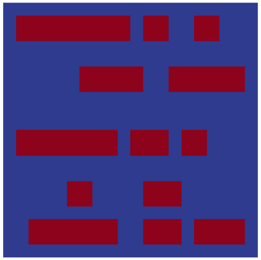
$$u_x = 0,\ \sigma_y = 0,\ x \in \Gamma_2 \cup \Gamma_4,$$

$$u_y = 0,\ \sigma_x = 0,\ x \in \Gamma_3, \quad \sigma_x = \sigma_y = 0, x \in \Gamma_1,$$

$$p = 0, \quad x \in \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5,$$

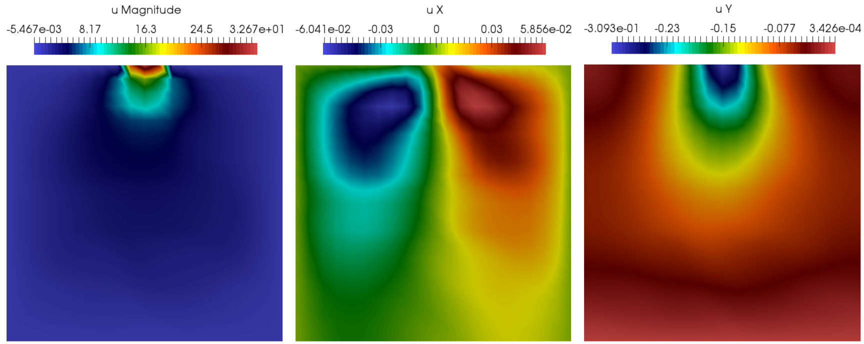
$$p_1 = 10, \quad x \in \Gamma_1.$$

Numerical results for 2D

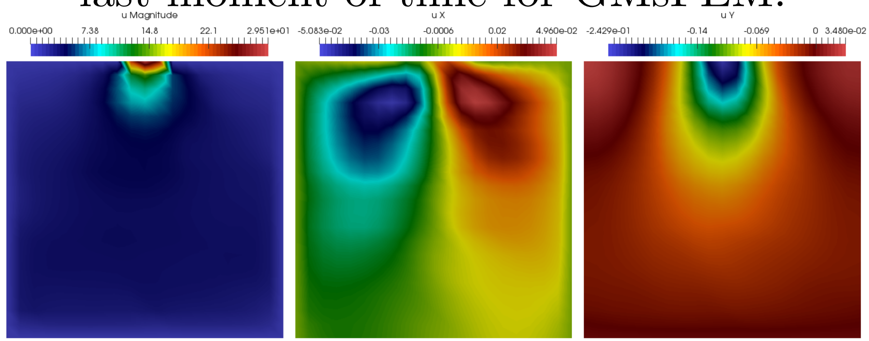


- $\lambda_1 = 5$ for domain with blue color $\lambda_2 = 10$ for domain with red color
- $\mu_1 = 4$ for domain with blue color $\mu_2 = 8$ for domain with red color
- $k_1 = 0.01$ for domain with blue color $k_2 = 0.05$ for domain with red color
- The calculation is performed by $T_{max} = 0.1$ with step in time $\tau = 0.01$
- Biot modulus - $\beta = 0.01$, Biot coefficient - $\alpha = 1$

Numerical results for 2D



Distribution of pressure, displacement along X and Y directions at the last moment of time for GMsFEM.



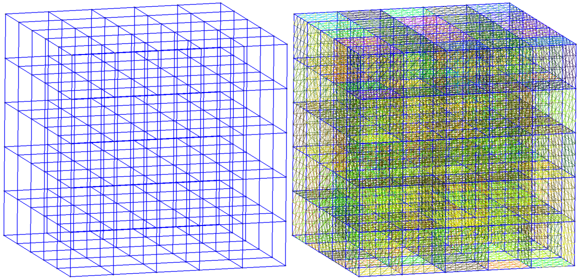
Distribution of pressure, displacement along X and Y directions at the last moment of time for fine grid.

Numerical results for 2D

L	L_2^u (%)	H_1^u (%)	L_2^p (%)	H_1^p (%)
1	95.845	96.098	97.017	99.633
2	40.431	56.988	61.624	66.461
4	13.940	23.854	22.094	39.869
8	1.031	5.596	2.240	12.950
12	0.063	0.561	0.062	0.514

Relative errors for displacement and pressure with different numbers of multiscale basis functions in GMsFEM

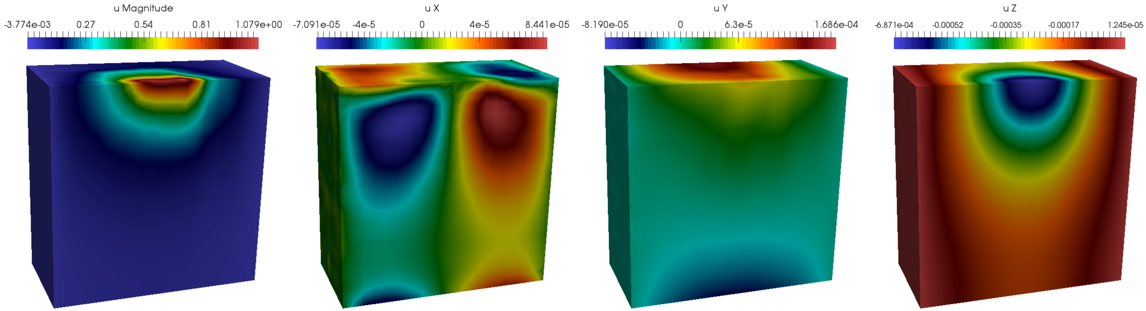
Numerical results for 3D



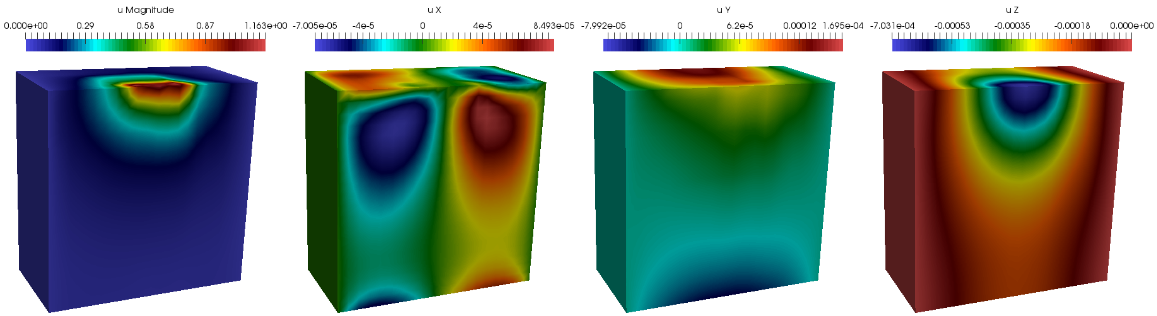
Geometric domain in a tetrahedral computational mesh.

- $\lambda_1 = 20$ for domain with blue color $\lambda_2 = 100$ for domain with red color
- $\mu_1 = 100$ for domain with blue color $\mu_2 = 200$ for domain with red color
- $k_1 = 1$ for domain with blue color $k_2 = 10$ for domain with red color
- The calculation is performed by $T_{max} = 0.001$ with step in time $\tau = 0.0001$
- Biot modulus - $\beta = 0.01$, Biot coefficient - $\alpha = 1$

Numerical results for 3D



The distribution of pressure and displacement along X, Y and Z (from left to right) at the final time for GMsFEM.



The distribution of pressure and displacement along X, Y and Z (from left to right) at the final time for fine grid.

Numerical results for 3D

L	L_2^u (%)	H_1^u (%)	L_2^p (%)	H_1^p (%)
1	17.423	43.768	28.158	53.484
2	14.137	35.863	18.698	43.129
4	5.437	20.492	9.294	31.934
8	3.124	14.134	4.306	21.463
12	1.643	9.635	2.369	16.109

Relative errors for displacement and pressure with different numbers of multiscale basis functions in GMsFEM for three-dimensional formulation.

Problem formulation in fractured media

We suppose that $\rho = const$ and $b = const$. Therefore, we have the following coupled system of equations for displacements, pressure in porous matrix and fractures

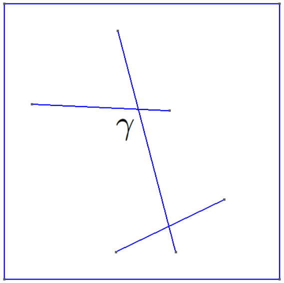
$$-\operatorname{div} \sigma(u) + \alpha \operatorname{grad} p = 0, \quad x \in \Omega,$$

$$c_m \frac{\partial p}{\partial t} + \alpha \frac{\partial \varepsilon_v}{\partial t} - \operatorname{div}\left(a_m \operatorname{grad} p\right) + \sigma_{mf}(p - p_f) = f, \quad x \in \Omega,$$

$$c_f \frac{\partial p_f}{\partial t} - \operatorname{div}(a_f \operatorname{grad} p_f) - \sigma_{fm}(p - p_f) = f_f, \quad x \in \gamma,$$

where

$$c_m = \frac{1}{M}, \quad c_f = \frac{b}{M_f}, \quad a_m = \frac{k}{\nu_f}, \quad a_f = b\frac{k_f}{\nu_f}.$$



Fine grid approximation

For numerical solution of the poroelasticity problem on fine grid use a standard finite element method. Find $(u, p) \in V \times Q$ such that

$$a_u(u, v) + b(p, v) = 0, \forall\ v \in \hat{V},$$
$$d\left(\frac{du}{dt}, q\right) + s\left(\frac{dp}{dt}, q\right) + a_p(p, q) = l(q), \quad \forall\ q \in \hat{Q}.$$

where $V = [H^1(\Omega)]^d$ and $Q = H^1(\Omega)$.

$$a_u(u, v) = \int_\Omega \sigma(u)\, \epsilon(v) dx, \quad l(q) = \int_\Omega f\, q\, dx + \sum_j \int_{\gamma j} f_f q dx,$$

$$a_p(p, q) = \int_\Omega \left(\frac{k}{\nu} \operatorname{grad} p, \operatorname{grad} q\right) dx + \sum_j \int_{\gamma j} \left(b \frac{k_f}{\nu_f} \operatorname{grad} p, \operatorname{grad} q\right) dx$$

$$c(p, q) = \int_\Omega \frac{1}{M}\, p\, q\, dx \ + \sum_j \int_{\gamma j} \frac{b}{M_f} p\, q\, dx,$$

$$b(p, v) = \int_\Omega \alpha(\operatorname{grad} p, v) dx, \quad d(u, q) = \int_\Omega \alpha \operatorname{div} u\, q\, dx.$$

Fine grid approximation

The standard implicit finite difference scheme is used for the time approximation of the pressure equation and we solve following coupled system in the matrix form on the fine grid.

$$\frac{1}{\tau}\begin{pmatrix} M & D \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p-\hat{p} \\ u-\hat{u} \end{pmatrix} + \begin{pmatrix} A & 0 \\ B & K \end{pmatrix}\begin{pmatrix} p \\ u \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix},$$

where $M = [m_{ij}]$, $A = [a_{ij}]$, $K = [k_{ij}]$, $D = [d_{ij}]$, $B = [b_{ij}]$,
$F = [f_j], M = M_m + M_f, A = A_m + A_f$

$$m_{ij} = \int_\Omega c_m \phi_i \, \phi_j \, dx + \sum_l \int_{\gamma l} c_f \psi_i \, \psi_j dx,$$

$$a_{ij} = \int_\Omega a_m \operatorname{grad} \phi_i \cdot \operatorname{grad} \phi_j \, dx \; + \sum_l \int_{\gamma l} a_f \operatorname{grad} \psi_i \, \psi_j \, dx,$$

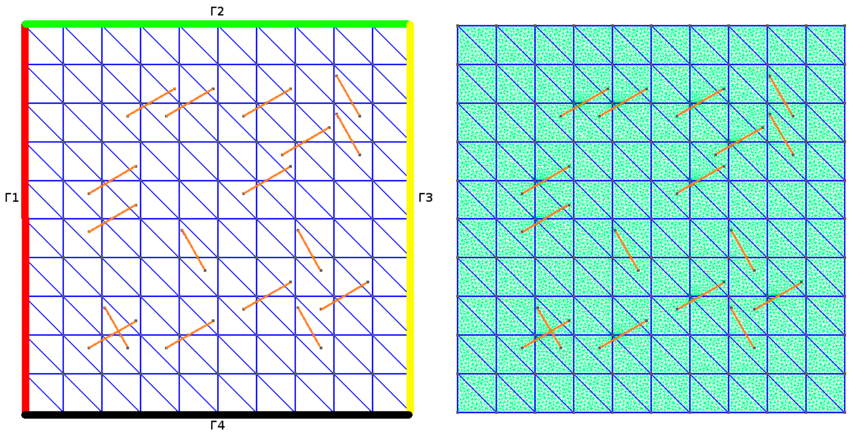
$$k_{ij} = \int_\Omega \sigma(\Phi_i) : \epsilon(\Phi_j) \, dx, \quad d_{ij} = \int_\Omega \alpha \operatorname{grad} \phi_i \, \Phi_j \, dx,$$

$$b_{ij} = \int_\Omega \alpha \operatorname{div} \Phi_i \, \phi_j \, dx, \quad f_j = \int_\Omega f \, \phi_j \, dx + \sum_l \int_{\gamma l} f_f \psi_j dx,$$

and Φ_i and ϕ_i are the linear basis functions for displacements and pressure.

Numerical results in fractured media

homogeneous media



Computational meshes. Left: coarse grid, Right: fine grid.

The numerical solution is presented for the following boundary conditions

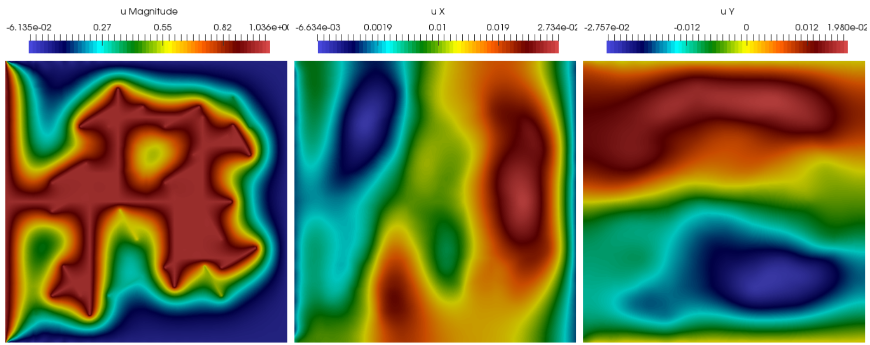
$$u_x = 0,\ \sigma_y = 0,\ x \in \Gamma_3 \cup \Gamma_1$$

$$u_y = 0,\ \sigma_x = 0,\ x \in \Gamma_4 \cup \Gamma_2$$

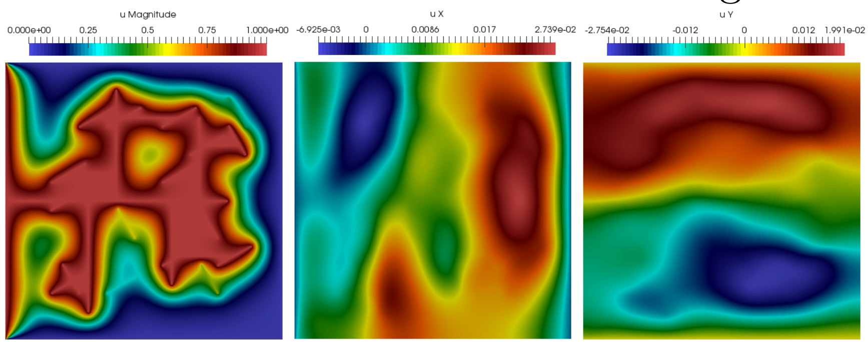
$$p = 1,\ x \in \Gamma_1.$$

Numerical results in fractured homogeneous media

homogeneous media



Distribution of pressure, displacement along X and Y directions at the last moment of time for GMsFEM in homogeneous media.



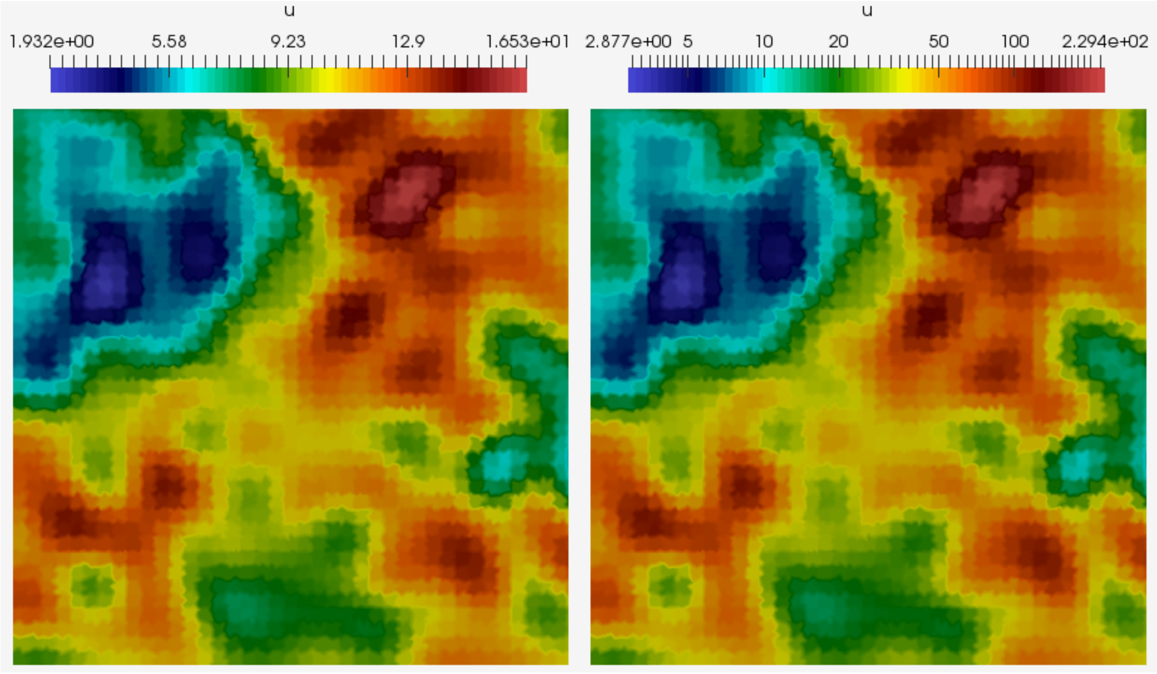
Distribution of pressure, displacement along X and Y directions at the last moment of time for fine grid in homogeneous media.

L	L_2^u (%)	H_1^u (%)	L_2^p (%)	H_1^p (%)
1	81.7277	77.8344	51.7067	127.035
2	30.8361	49.2153	29.7308	93.3461
4	7.31921	14.7866	6.71832	39.985
8	2.3319	5.65635	0.985431	12.5086
12	1.69256	5.59024	0.339608	7.26006

Relative errors for displacement and pressure with different numbers of multiscale basis functions for GMsFEM in homogeneous media.

Numerical results in fractured heterogeneous media

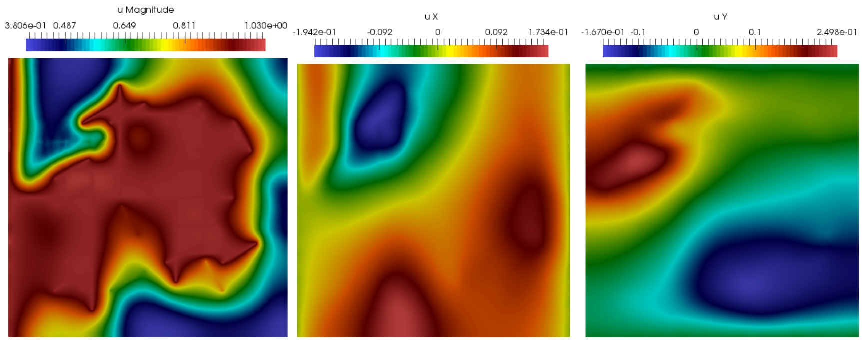
heterogeneous media



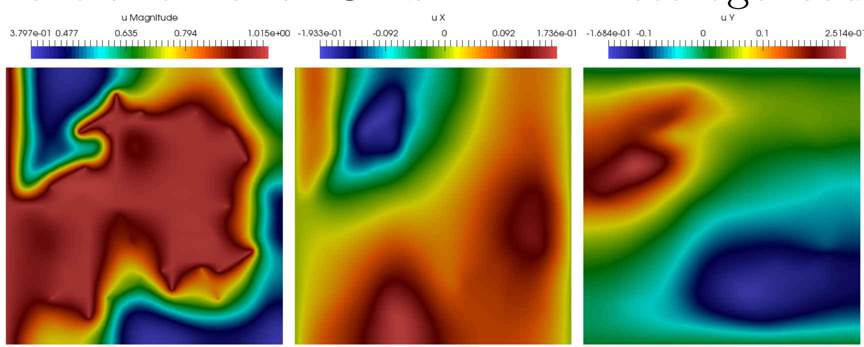
Elasticity parameter E and heterogeneous permeability.

Numerical results in fractured heterogeneous media

heterogeneous media



Distribution of pressure, displacement along X and Y directions at the last moment of time for GMsFEM in heterogeneous media.



Distribution of pressure, displacement along X and Y directions at the last moment of time for GMsFEM in heterogeneous media.

L	L_2^u (%)	H_1^u (%)	L_2^p (%)	H_1^p (%)
1	80.9641	69.2054	14.951	141.648
2	23.159	41.3603	11.5642	102.523
4	11.1475	14.2097	2.71181	34.5921
8	5.63175	10.4813	3.60061	29.3502
12	1.14291	5.16293	0.460655	6.42927

Relative errors for displacement and pressure with different numbers of multiscale basis functions for GMsFEM in heterogeneous media.

Conclusion

- We considered the coupled system of equations for pressure and displacements in fractured media.
- For approximation of the problem on the fine grid the finite element method is used.
- For coarse grid approximation, the Generalized Multiscale Finite Element Methods for the poroelasticity problem in fractured heterogeneous porous media is presented.
- We calculate the relative errors for different number of the multiscale basis functions.
- We observe that the presented method can provide good accuracy in homogeneous and heterogeneous domains.

Thank you for your attention!