The Marcinkiewicz-type discretization theorems

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1 Introduction

2 The Marcinkiewicz-type theorems

Difinition

We are interested in discretizing the L_q, $1 \leq q \leq \infty$, norm of elements of an N-dimensional subspace X_N. We call such results the Marcinkiewicz-type discretization theorems. There are different settings and different ingredients, which play important role in this problem.

Marcinkiewicz problem

Let Ω be a compact subset of $\mathbb{R}^d$ with the probability measure μ. We say that a linear subspace X_N of the $L_q(\Omega)$, $1 \le q < \infty$, admits the Marcinkiewicz-type discretization theorem with parameters m and q if there exist a set $\{\xi^\nu \in \Omega, \nu = 1, \dots, m\}$ and two positive constants $C_j(d,q)$, $j = 1,2$, such that for any $f \in X_N$ we have

$$C_1(d,q)\|f\|_q^q \le \frac{1}{m}\sum_{\nu=1}^{m} |f(\xi^\nu)|^q \le C_2(d,q)\|f\|_q^q. \qquad (1)$$

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In the case $q = \infty$ we define L_∞ as the space of continuous on Ω functions and ask for

$$C_1(d)\|f\|_\infty \le \max_{1\le\nu\le m} |f(\xi^\nu)| \le \|f\|_\infty. \qquad (2)$$

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We will also use a brief way to express the above property: the $\mathcal{M}(m,q)$ theorem holds for a subspace X_N or $X_N \in \mathcal{M}(m,q)$.

Marcinkiewicz problem with weights

We say that a linear subspace X_N of the $L_q(\Omega)$, $1 \le q < \infty$, admits the weighted Marcinkiewicz-type discretization theorem with parameters m and q if there exist a set of knots $\{\xi^\nu \in \Omega\}$, a set of weights $\{\lambda_\nu\}$, $\nu = 1, \dots, m$, and two positive constants $C_j(d, q)$, $j = 1, 2$, such that for any $f \in X_N$ we have

$$C_1(d,q)\|f\|_q^q \le \sum_{\nu=1}^{m} \lambda_\nu |f(\xi^\nu)|^q \le C_2(d,q)\|f\|_q^q. \qquad (3)$$

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$$C_1(d,q)\|f\|_q^q \le \sum_{\nu=1}^{m} \lambda_\nu |f(\xi^\nu)|^q \le C_2(d,q)\|f\|_q^q. \qquad (3)$$

Then we also say that the $\mathcal{M}^w(m, q)$ theorem holds for a subspace X_N or $X_N \in \mathcal{M}^w(m, q)$. Obviously, $X_N \in \mathcal{M}(m, q)$ implies that $X_N \in \mathcal{M}^w(m, q)$.

Marcinkiewicz problem with ε

We write $X_N \in \mathcal{M}(m,q,\varepsilon)$ if (1) holds with $C_1(d,q) = 1-\varepsilon$ and $C_2(d,q) = 1+\varepsilon$. Respectively, we write $X_N \in \mathcal{M}^w(m,q,\varepsilon)$ if (3) holds with $C_1(d,q) = 1-\varepsilon$ and $C_2(d,q) = 1+\varepsilon$.

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We note that the most powerful results are for $\mathcal{M}(m,q,0)$, when the L_q norm of $f \in X_N$ is discretized exactly by the formula with equal weights $1/m$.

Some remarks for the case $q = 2$

We describe the properties of the subspace X_N in terms of a system $\mathcal{U}_N := \{u_i\}_{i=1}^N$ of functions such that $X_N = \operatorname{span}\{u_i, i = 1, \dots, N\}$. In the case $X_N \subset L_2$ we assume that the system is orthonormal on Ω with respect to measure μ. In the case of real functions we associate with $x \in \Omega$ the matrix $G(x) := [u_i(x)u_j(x)]_{i,j=1}^N$. Clearly, $G(x)$ is a symmetric positive semi-definite matrix of rank 1. It is easy to see that for a set of points $\xi^k \in \Omega$, $k = 1, \dots, m$, and $f = \sum_{i=1}^N b_i u_i$ we have

$$\sum_{k=1}^m \lambda_k f(\xi^k)^2 - \int_\Omega f(x)^2 d\mu = \mathbf{b}^T \left(\sum_{k=1}^m \lambda_k G(\xi^k) - I \right) \mathbf{b},$$

where $\mathbf{b} = (b_1, \dots, b_N)^T$ is the column vector and I is the identity matrix.

Remarks continue

Therefore, the $\mathcal{M}^w(m,2)$ problem is closely connected with a problem of approximation (representation) of the identity matrix I by an m-term approximant with respect to the system $\{G(x)\}_{x\in\Omega}$. It is easy to understand that under our assumptions on the system $\mathcal{U}_N$ there exist a set of knots $\{\xi^k\}_{k=1}^m$ and a set of weights $\{\lambda_k\}_{k=1}^m$, with $m\le N^2$ such that

$$I=\sum_{k=1}^m \lambda_k G(\xi^k)$$

and, therefore, we have for any $X_N\subset L_2$ that

$$X_N\in\mathcal{M}^w(N^2,2,0).$$

Condition E

We begin with formulation of the Rudelson result from 1999. Let $\Omega_M = \{x^j\}_{j=1}^M$ be a discrete set with the probability measure $\mu(x^j) = 1/M$, $j = 1, \dots, M$. Assume that $\{u_i(x)\}_{i=1}^N$ is a real orthonormal on Ω_M system satisfying the following condition: **E** for all j

$$\sum_{i=1}^{N} u_i(x^j)^2 \leq Nt^2$$

with some $t \geq 1$.

Rudelson's theorem

Then for every $\epsilon > 0$ there exists a set $J \subset \{1, \dots, M\}$ of indices with cardinality

$$m := |J| \le C \frac{t^2}{\epsilon^2} N \log \frac{Nt^2}{\epsilon^2}$$

such that for any $f = \sum_{i=1}^{N} c_i u_i$ we have

$$(1-\epsilon)\|f\|_2^2 \le \frac{1}{m} \sum_{j \in J} f(x^j)^2 \le (1+\epsilon)\|f\|_2^2.$$

A slight improvement

Theorem (VT, 2017)

Let $\{u_i\}_{i=1}^N$ be an orthonormal system, satisfying condition **E**. *Then for every $\epsilon > 0$ there exists a set $\{\xi^j\}_{j=1}^m \subset \Omega$ with*

$$m \leq C\frac{t^2}{\epsilon^2} N \log N$$

such that for any $f = \sum_{i=1}^N c_i u_i$ we have

$$(1-\epsilon)\|f\|_2^2 \leq \frac{1}{m}\sum_{j=1}^m f(\xi^j)^2 \leq (1+\epsilon)\|f\|_2^2.$$

The Marcinkiewicz-type theorem with weights

We now comment on a recent breakthrough result by J. Batson, D.A. Spielman, and N. Srivastava, 2012. We formulate their result in our notations. Let as above $\Omega_M = \{x^j\}_{j=1}^M$ be a discrete set with the probability measure $\mu(x^j) = 1/M$, $j = 1, \dots, M$. Assume that $\{u_i(x)\}_{i=1}^N$ is a real orthonormal on Ω_M system. Then for any number $d > 1$ there exist a set of weights $w_j \geq 0$ such that $|\{j : w_j \neq 0\}| \leq dN$ so that for any $f \in \text{span}\{u_1, \dots, u_N\}$ we have

$$\|f\|_2^2 \leq \sum_{j=1}^{M} w_j f(x^j)^2 \leq \frac{d+1+2\sqrt{d}}{d+1-2\sqrt{d}} \|f\|_2^2.$$

A comment

The proof of this result is based on a delicate study of the m-term approximation of the identity matrix I with respect to the system $\mathcal{D} := \{G(x)\}_{x\in\Omega}$, $G(x) := [u_i(x)u_j(x)]_{i,j=1}^N$ in the spectral norm. The authors control the change of the maximal and minimal eigenvalues of a matrix, when they add a rank one matrix of the form $wG(x)$. Their proof provides an algorithm for construction of the weights $\{w_j\}$. In particular, this implies that

$$X_N(\Omega_M) \in \mathcal{M}^w(m,2,\epsilon) \quad \text{provided} \quad m \geq CN\epsilon^{-2}$$

with large enough C.

Further results

Theorem (1; VT, 2017)

Let $\Omega_M = \{x^j\}_{j=1}^M$ *be a discrete set with the probability measure* $\mu(x^j) = 1/M$, $j = 1, \dots, M$. *Assume that* $\{u_i(x)\}_{i=1}^N$ *is an orthonormal on* Ω_M *system (real or complex). Assume in addition that this system has the following property: for all* $j = 1, \dots, M$ *we have* $\sum_{i=1}^N |u_i(x^j)|^2 = N$. *Then there is an absolute constant* C_1 *such that there exists a subset* $J \subset \{1, 2, \dots, M\}$ *with the property:* $m := |J| \le C_1 N$ *and for any* $f \in X_N := \operatorname{span}\{u_1, \dots, u_N\}$ *we have*

$$C_2 \|f\|_2^2 \le \frac{1}{m} \sum_{j \in J} |f(x^j)|^2 \le C_3 \|f\|_2^2,$$

where C_2 *and* C_3 *are absolute positive constants.*

NOU Lemma

The above theorem is based on the following lemma from S. Nitzan, A. Olevskii, and A. Ulanovskii, 2016.

Lemma (NOU, 2016)

Let a system of vectors $\mathbf{v}_1,\dots,\mathbf{v}_M$ *from* $\mathbb{C}^N$ *have the following properties: for all* $\mathbf{w}\in\mathbb{C}^N$ *we have* $\sum_{j=1}^M |\langle \mathbf{w},\mathbf{v}_j\rangle|^2 = \|\mathbf{w}\|_2^2$ *and* $\|\mathbf{v}_j\|_2^2 = N/M, \quad j=1,\dots,M$. *Then there is a subset* $J\subset\{1,2,\dots,M\}$ *such that for all* $\mathbf{w}\in\mathbb{C}^N$

$$c_0\|\mathbf{w}\|_2^2 \le \frac{M}{N}\sum_{j\in J}|\langle \mathbf{w},\mathbf{v}_j\rangle|^2 \le C_0\|\mathbf{w}\|_2^2,$$

where c_0 *and* C_0 *are some absolute positive constants.*

Fundamental theorem

The above Lemma was derived from the following theorem from A. Marcus, D.A. Spielman, and N. Srivastava, 2015, which solved the **Kadison-Singer problem**.

Theorem (MSS, 2015)

Let a system of vectors $\mathbf{v}_1, \dots, \mathbf{v}_M$ *from* $\mathbb{C}^N$ *have the following properties: for all* $\mathbf{w} \in \mathbb{C}^N$ *we have* $\sum_{j=1}^M |\langle \mathbf{w}, \mathbf{v}_j \rangle|^2 = \|\mathbf{w}\|_2^2$ *and* $\|\mathbf{v}_j\|_2^2 \leq \epsilon$.
Then there exists a partition of $\{1, \dots, M\}$ *into two sets* S_1 *and* S_2, *such that for each* $i = 1, 2$ *we have for all* $\mathbf{w} \in \mathbb{C}^N$

$$\sum_{j \in S_i} |\langle \mathbf{w}, \mathbf{v}_j \rangle|^2 \leq \frac{(1 + \sqrt{2\epsilon})^2}{2} \|\mathbf{w}\|_2^2.$$

Trigonometric polynomials

Let Q be a finite subset of $\mathbb{Z}^d$. We denote

$$\mathcal{T}(Q) := \{f : f = \sum_{\mathbf{k}\in Q} c_{\mathbf{k}} e^{i(\mathbf{k},\mathbf{x})}\}.$$

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The above Theorem (1; VT, 2017) implies the following result.

Theorem (2; VT, 2017)

There are three positive absolute constants C_1, C_2, and C_3 with the following properties: For any $d \in \mathbb{N}$ and any $Q \subset \mathbb{Z}^d$ there exists a set of $m \le C_1|Q|$ points $\xi^j \in \mathbb{T}^d$, $j = 1, \dots, m$ such that for any $f \in \mathcal{T}(Q)$ we have

$$C_2\|f\|_2^2 \le \frac{1}{m}\sum_{j=1}^{m} |f(\xi^j)|^2 \le C_3\|f\|_2^2.$$

Conditions on X_N

We now proceed to the L_1 case. We impose the following assumptions on the system $\{u_i\}_{i=1}^N$ of real functions.
A. There exist $\alpha > 0$, β, and K_1 such that for all $i \in [1, N]$ we have

$$|u_i(\mathbf{x}) - u_i(\mathbf{y})| \le K_1 N^\beta \|\mathbf{x} - \mathbf{y}\|_\infty^\alpha, \quad \mathbf{x}, \mathbf{y} \in \Omega. \tag{4}$$

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B. There exists a constant K_2 such that $\|u_i\|_\infty^2 \le K_2$, $i = 1, \dots, N$.

C. Denote $X_N := \text{span}(u_1, \dots, u_N)$. There exist two constants K_3 and K_4 such that the following Nikol'skii-type inequality holds for all $f \in X_N$

$$\|f\|_\infty \le K_3 N^{K_4/p} \|f\|_p, \quad p \in [2, \infty). \tag{5}$$

Main theorem

Theorem (3; VT, 2017)

Suppose that a real orthonormal system $\{u_i\}_{i=1}^N$ *satisfies conditions* **A**, **B**, *and* **C**. *Then there exists a set of* $m \le C_1 N(\log N)^{7/2}$ *points* $\xi^j \in \Omega$, $j = 1, \dots, m$, $C_1 = C(d, K_1, K_2, K_3, K_4, \Omega, \alpha, \beta)$, *such that for any* $f \in X_N$ *we have*

$$\frac{1}{2}\|f\|_1 \le \frac{1}{m}\sum_{j=1}^{m} |f(\xi^j)| \le \frac{3}{2}\|f\|_1.$$

Definition of the entropy numbers

Let X be a Banach space and let B_X denote the unit ball of X with the center at 0. Denote by $B_X(y,r)$ a ball with center y and radius r: $\{x \in X : \|x-y\| \le r\}$. For a compact set A and a positive number ε we define the covering number $N_\varepsilon(A,X)$ as follows

$$N_\varepsilon(A,X) := \min\{n : \exists y^1,\dots,y^n, y^j \in A : A \subseteq \cup_{j=1}^n B_X(y^j,\varepsilon)\}.$$

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It is convenient to consider along with the entropy $H_\varepsilon(A,X) := \log_2 N_\varepsilon(A,X)$ the entropy numbers $\varepsilon_k(A,X)$:

$$\varepsilon_k(A,X) := \inf\{\varepsilon : \exists y^1,\dots,y^{2^k} \in A : A \subseteq \cup_{j=1}^{2^k} B_X(y^j,\varepsilon)\}.$$

In our definition of $N_\varepsilon(A,X)$ and $\varepsilon_k(A,X)$ we require $y^j \in A$. In a standard definition of $N_\varepsilon(A,X)$ and $\varepsilon_k(A,X)$ this restriction is not imposed. However, it is well known that these characteristics may differ at most by a factor 2.

Conditional theorem

Theorem (4; VT2017)

Suppose that a real N-dimensional subspace X_N satisfies the following condition on the entropy numbers of the unit ball $X_N^1 := \{f \in X_N : \|f\|_1 \le 1\}$ with $B \ge 1$

$$\varepsilon_k(X_N^1, L_\infty) \le B \begin{cases} N/k, & k \le N, \\ 2^{-k/N}, & k \ge N. \end{cases}$$

Then there exists a set of $m \le C_1 NB(\log_2(2N\log_2(8B)))^2$ points $\xi^j \in \Omega$, $j = 1, \dots, m$, with large enough absolute constant C_1, such that for any $f \in X_N$ we have

$$\frac{1}{2}\|f\|_1 \le \frac{1}{m}\sum_{j=1}^{m} |f(\xi^j)| \le \frac{3}{2}\|f\|_1.$$

Hyperbolic cross polynomials

For $\mathbf{s} \in \mathbb{Z}_+^d$ define

$$\rho(\mathbf{s}) := \{\mathbf{k} \in \mathbb{Z}^d : [2^{s_j-1}] \le |k_j| < 2^{s_j}, \quad j = 1, \dots, d\}$$

where $[x]$ denotes the integer part of x.

Hyperbolic cross polynomials

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where $[x]$ denotes the integer part of x. We define the step hyperbolic cross Q_n as follows

$$Q_n := \cup_{\mathbf{s}:\|\mathbf{s}\|_1 \le n} \rho(\mathbf{s})$$

and the corresponding set of the hyperbolic cross polynomials as

$$\mathcal{T}(Q_n) := \{f : f = \sum_{\mathbf{k} \in Q_n} c_{\mathbf{k}} e^{i(\mathbf{k},\mathbf{x})}\}.$$

Discretization for the hyperbolic cross polynomials

Theorem (5; VT, 2017)

Let $d \in \mathbb{N}$. *For any* $n \in \mathbb{N}$ *there exists a set of* $m \le C_1(d)|Q_n|n^{7/2}$ *points* $\xi^j \in \mathbb{T}^d$, $j = 1, \dots, m$ *such that for any* $f \in \mathcal{T}(Q_n)$ *we have*

$$C_2(d)\|f\|_1 \le \frac{1}{m}\sum_{j=1}^{m} |f(\xi^j)| \le C_3(d)\|f\|_1.$$

General trigonometric polynomials

Theorem (6; VT, 2017)

For any $Q \subset \Pi(\mathbf{N})$ *with* $\mathbf{N} = (2^n, \dots, 2^n)$ *and* $\epsilon \in [2^{1-2^{nd/2}}, 1/2]$ *there exists a set of* $m \leq C_1(d)|Q|n^{7/2}\epsilon^{-2}$ *points* $\xi^j \in \mathbb{T}^d$, $j = 1, \dots, m$ *such that for any* $f \in \mathcal{T}(Q)$ *we have*

$$(1-\epsilon)\|f\|_1 \leq \frac{1}{m}\sum_{j=1}^{m}|f(\xi^j)| \leq (1+\epsilon)\|f\|_1.$$