Generalized Multiscale Finite Element Method for unsaturated filtration problem in fractured medium

D. Spiridonov

Multiscale model reduction Laboratory, North-Eastern Federal University, Yakutsk, Russia.
joint work with M. Vasilyeva, Eric T. Chung, Y. Efendiev

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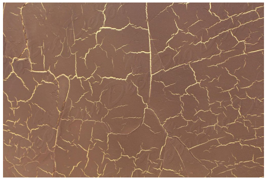
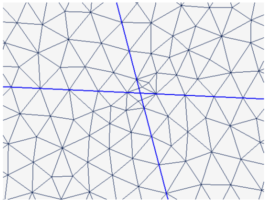
Inctroduction

- Fractured porous media are characterized by the presence of fractures at multiple scales
- To present the microscale interaction between the fractures and the matrix, various coarse grid models have been developed. These include dual-continua approaches, coarse-scale continuum model, upscaling methods, Multiscale Finite Volume, and so on.
- To present the microscale interaction between the fractures and the matrix, we use Generalized Multiscale Finite Element Method (GMsFEM)
- The main idea is to use multiscale basis functions to extract an essential information in each coarse region and develop a local reduced order model.

Problem formulation

The water flow into the porous media is driven by the pressure gradient and describes by the Darcy's Law

$$q = k(x, p)\, \mathrm{grad}(p + z),$$

where q is the velocity vector, k is the unsaturated hydraulic conductivity tensor and z represent the influence of the gravity to the flow.
For the fluid flow in domain Ω, we have following equation

$$\frac{\partial \Theta}{\partial t} + \mathrm{div}\, q = 0, \quad x \in \Omega,$$

where Θ is the water content and represents the fraction of porous medium total volume that is filled with fluid.

Problem formulation

Havercamp model

As constitutive relations, we use Havercamp model

$$\Theta(p) = \frac{\alpha(\Theta_s - \Theta_r)}{\alpha + |p|^\beta} + \Theta_r,$$

$$k_m(x,p) = k_s(x)\frac{A}{A + |p|^\epsilon} + \Theta_r,$$

where $k_s(x)$ is also known as the saturated hydraulic conductivity. We use following initial condition

$$p(x) = p_0, \quad x \in \Omega, \quad t = 0,$$

and boundary condition

$$p(x) = p_1, \quad x \in \Gamma_D,$$

$$q \cdot n = 0, \quad x \in \Gamma_N.$$

where $\partial\Omega = \Gamma_D \cup \Gamma_N$.

Problem formulation

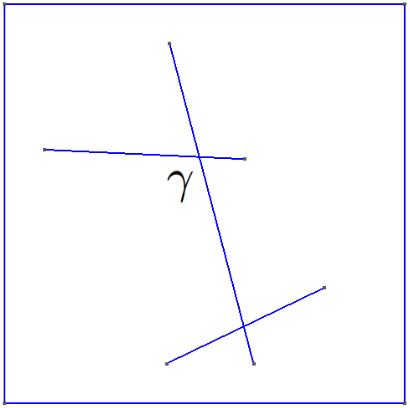
Discrete Fracture Model

We have following problem in domain Ω

$$\begin{cases} \frac{\partial \Theta_m}{\partial t} - \operatorname{div}\left(k_m(x, p_m)(\operatorname{grad} p_m - z)\right) + \sigma_{mf}(p_m - p_f) = F_m, & x \in \Omega, \\ \frac{\partial \Theta_f}{\partial t} - \operatorname{div}\left(k_f(x, p_f)(\operatorname{grad} p_f - z)\right) - \sigma_{fm}(p_m - p_f) = F_f, & x \in \gamma, \end{cases}$$

$$\int_\Omega \frac{\partial \Theta_m}{\partial t} v_m \, dx + \int_\Omega (k_m \operatorname{grad} p_m, \operatorname{grad} v_m) \, dx - \int_\Omega \frac{\partial k_m}{\partial z} v_m dx + \\ + \int_\Omega \sigma(p_m - p_f) \, v_m dx = \int_\Omega F_m \, v_m dx,$$

$$\int_\gamma \frac{\partial \Theta_f}{\partial t} v_f \, dx + \int_\gamma (k_f \operatorname{grad} p_f, \operatorname{grad} v_f) \, dx - \int_\gamma \frac{\partial k_f}{\partial z} v_f dx - \\ - \int_\gamma \sigma(p_m - p_f) \, v_f dx = \int_\gamma F_f \, v_f dx.$$



To approximate nonlinear coeficients we use simplified approximation from previous time step

$$\int_\Omega \frac{\Theta_m^{n+1} - \Theta_m^n}{\tau} v_m \, dx + \int_\Omega (k_m^n \operatorname{grad} p_m^{n+1}, \operatorname{grad} v_m) \, dx - \int_\Omega \frac{\partial k_m^n}{\partial z} v_m dx + \\ + \int_\Omega \sigma^n (p_m^{n+1} - p_f^{n+1}) \, v_m dx = \int_\Omega F_m \, v_m dx,$$

$$\int_\Omega \frac{\Theta_f^{n+1} - \Theta_f^n}{\tau} v_f \, dx + \int_\Omega \left(k_f^n \operatorname{grad} p_f^{n+1}, \operatorname{grad} v_f\right) dx - \int_\Omega \frac{\partial k_f^n}{\partial z} v_f dx - \\ - \int_\Omega \sigma^n (p_m^{n+1} - p_f^{n+1}) \, v_f dx = \int_\Omega F_f \, v_f dx.$$

Fine grid approximation

Matrix form

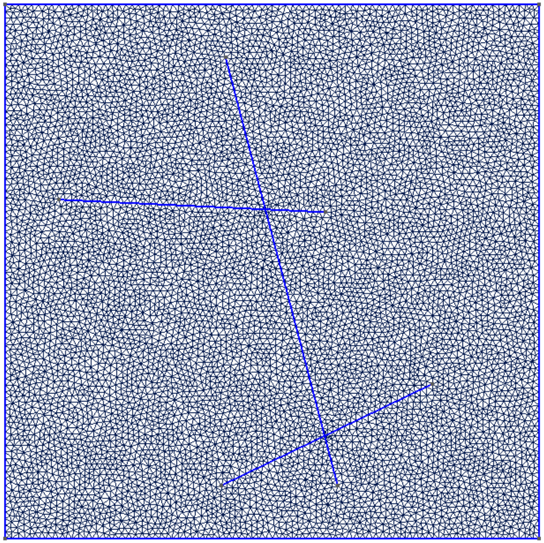
We can write approximation, in the matrix form as

$$\frac{1}{\tau}\begin{pmatrix} S_m & 0 \\ 0 & S_f \end{pmatrix}\begin{pmatrix} p_m - \check{p}_m \\ p_f - \check{p}_f \end{pmatrix} + \begin{pmatrix} A_m + Q & -Q \\ -Q & A_f + Q \end{pmatrix}\begin{pmatrix} p_m \\ p_f \end{pmatrix} = \begin{pmatrix} F_m \\ F_f \end{pmatrix}$$

We assume $p_m = p_f$ on Ω and using superposition principle, we obtain

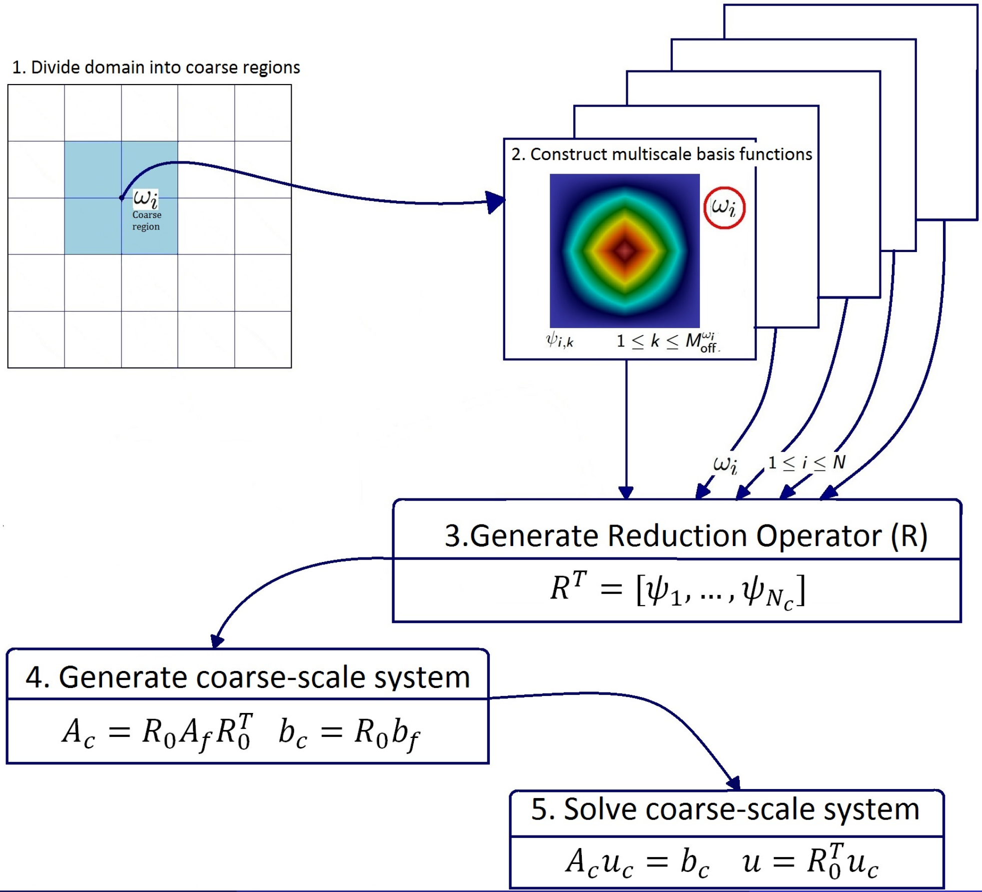
$$S\frac{p - \check{p}}{\tau} + Ap = F$$

where $S = S_m + S_f$, $A = A_m + A_f$ and $F = F_m + F_f$.
We built computational grid, which take into account fractures explicitly



Coarse grid approximation

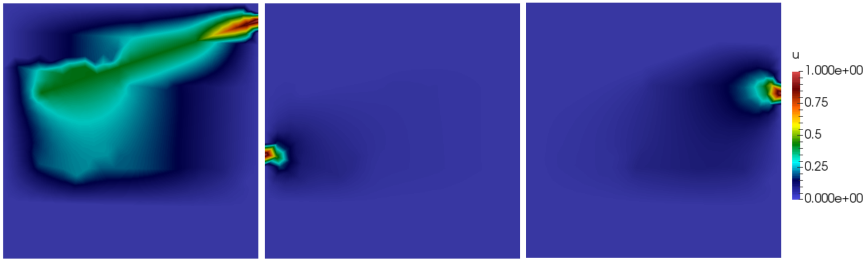
GMsFEM algorithm



In the multiscale basis calculations, we first construct a snapshot space $V_{\text{snap}}^{\omega_i}$. The snapshot space is constructed by the solution of the following local problems

$$-\operatorname{div}(k_s(x)\nabla\psi_l) = 0 \quad x \in \omega_i \tag{4.1}$$

with boundary conditions $\psi_l(x) = \delta_j$ on $\partial\omega_i$ and δ_j is the function, which takes the value 1 at $x = x_j$ and zero elsewhere.



Coarse grid approximation using GMsFEM

Spectral problem

Next, we solve a local spectral problems on the snapshot space

$$A\varphi^i = \lambda S \varphi^i,$$

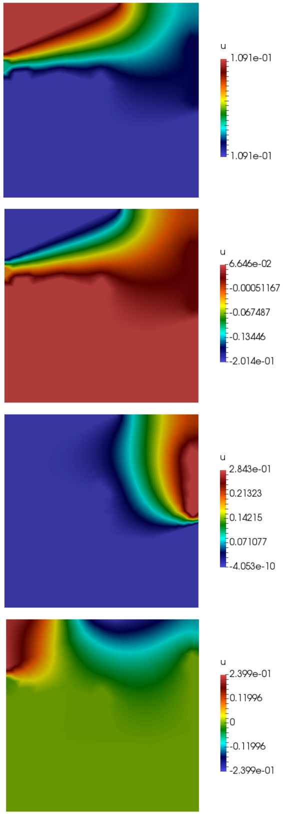
where the elements of the matrices $A = \{a_{ij}\}$ and $S = \{s_{ij}\}$ are defined as follow

$$a_{ij} = \int_{\omega_i} (k_s(x) \nabla u, \nabla q) dx, \quad s_{ij} = \int_{\omega_i} k_s(x)\, u\, q\, dx.$$

We make transition on the snapshot space

$$\tilde{A}\tilde{\varphi}^i = \lambda \tilde{S} \tilde{\varphi}^i, \quad \tilde{A} = PAP^T, \quad \text{and} \quad \tilde{S} = PSP^T.$$

where $P = \{\psi_0, \psi_1, ..., \psi_{J_i}\}$ and $\varphi^i_k = P^T \tilde{\varphi}^i_k$ for $k = 1, 2, ...$ Then, we choose the smallest M_i eigenvalues and use them for the construction of multiscale basis functions.



Coarse grid approximation

Multiscale basis functions

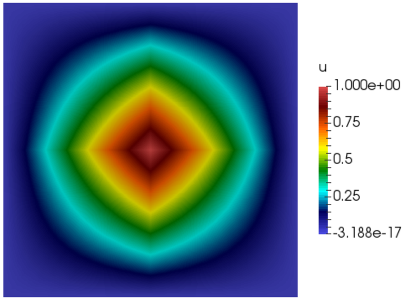
The multiscale space is defined as the span of $\chi_i \varphi_k^i$, where χ_i is the usual nodal basis function for the node i (linear partition of unity functions). The number of bases can be different, the accuracy of the solution can be improved when we increase the number of bases. Finally, we create following matrix for each ω_i

$$R^i = \left[\chi_i \varphi_1^i, \dots, \chi_i \varphi_{M_i}^i,\right].$$

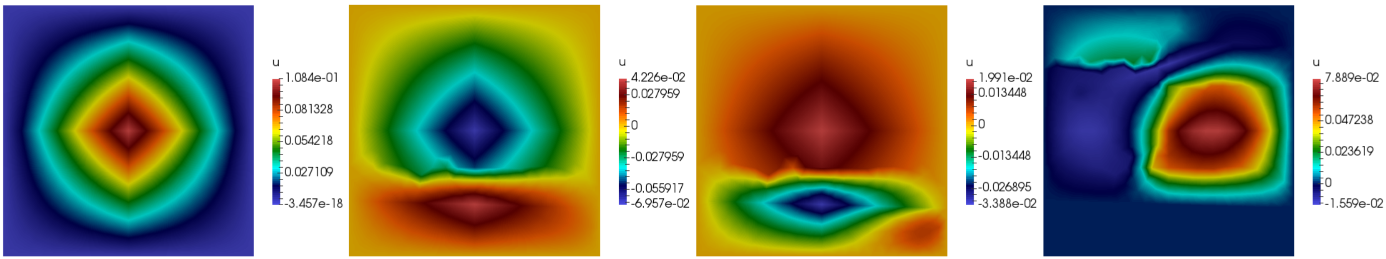
and define a transition matrix from a fine grid to a coarse grid to reduce the dimension of the problem

$$R = [R^1, R^2, ..., R^{N_v}],$$

where N_v is the number of local domains ω_i.



Coarse grid approximation



We have following coarse grid approximation

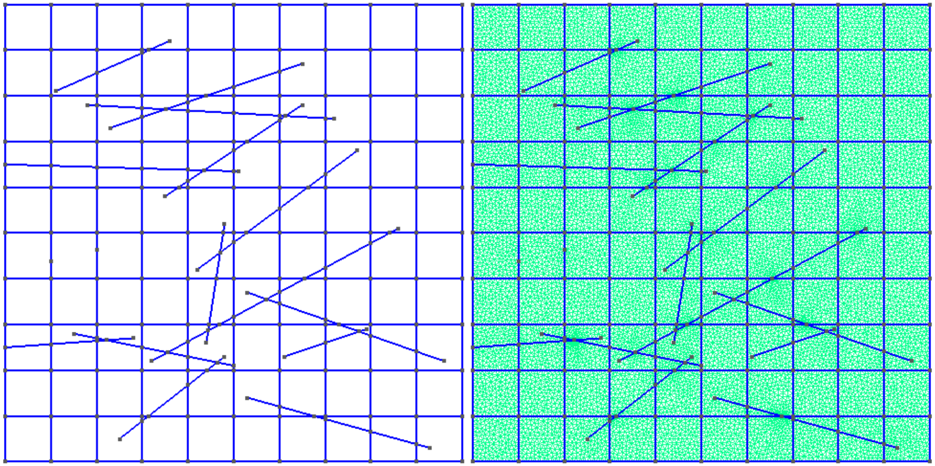
$$S_c^m \frac{p_H^{n+1,m+1} - p_H^{n+1,m}}{\tau} + A_c^m p_H^{n+1,m+1} = F_c^m,$$

where $S_c^m = R S^m R^T$, $A_c^m = R A^m R^T$ and $F_c^m = R F^m$. Here, using multiscale solution p_H, we can reconstruct a fine grid solution $p = R^T p_H$.

Numerical results

2D fractured medium

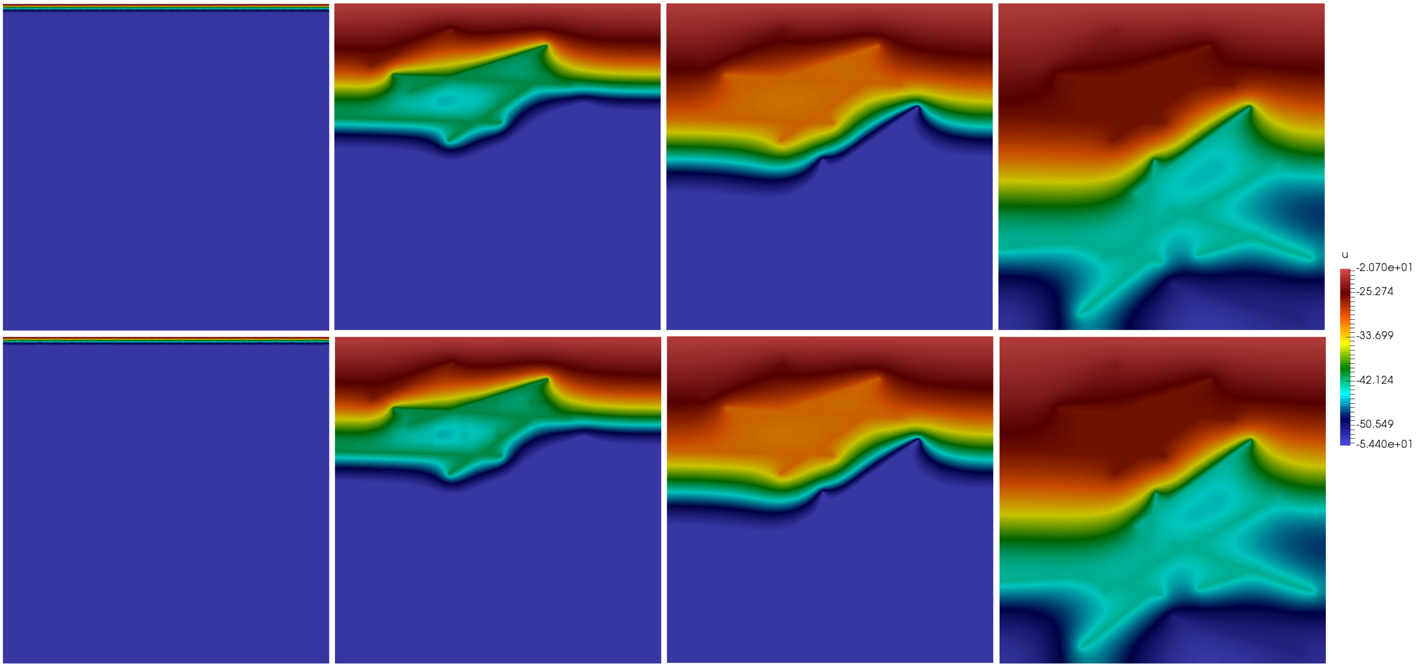
We consider water infiltration into porous medium which size 10 meters to 10 meters. As boundary conditions we use $p_1 = -20.7$ on top boundary Γ_1 and $p_0 = -61.5$ for initial conditions. For soil properties we use $\alpha = 1.511 \times 10^6$, $\beta = 3.96$, $\Theta_s = 0.287$, $\Theta_r = 0.075$, $A = 1.175 \times 10^6$, $F = 0$, $\gamma = 4.74$, $k_m = 0.1$ and $k_f = 10^9$ We took the maximum time equal to $14 \cdot 10^{-4}$ s. and 200 time layers.



Computational domain(left) and fine grid(right) with 31,5 thousands cells and 14,5.

Numerical results

2D heterogeneous medium



Fine scale solution(top line) and multsicale solution using 16 basis functions (bottom line). The results presented on 1, 50, 100, 200 time layers.

Numerical results

2D fractured medium. L_2 error

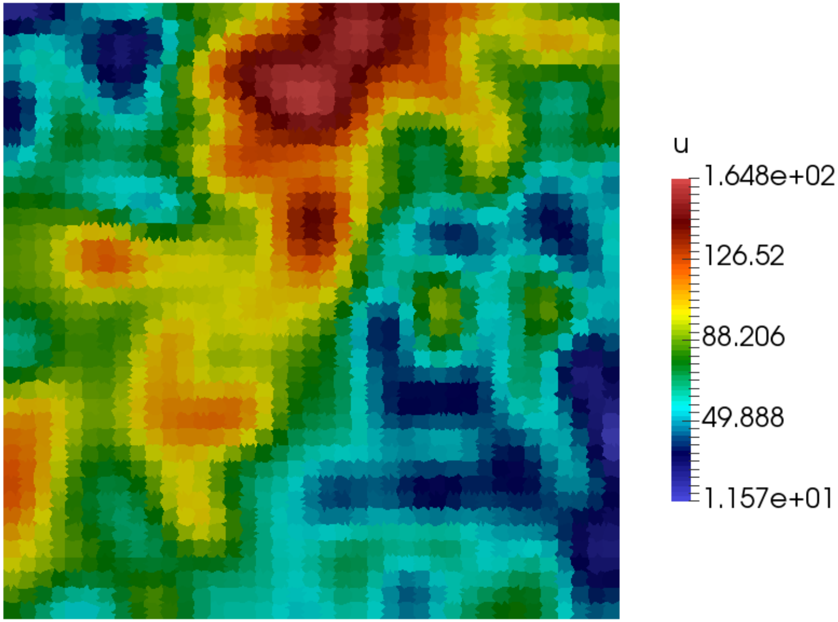
Number of bases	DOF_c	t_1	t_{50}	t_{100}	t_{200}
1	121	86.21	92.92	92.92	92.92
2	242	64.26	87.21	87.21	87.21
4	484	2.16	0.37	0.29	0.29
8	968	0.81	0.13	0.09	0.11
12	1452	0.36	0.07	0.06	0.07
16	1936	0.22	0.06	0.05	0.04

L_2 error for different number of basis functions. The size of fine grid system is 14.5 thousands.

Numerical results

2D heterogeneous fractured medium

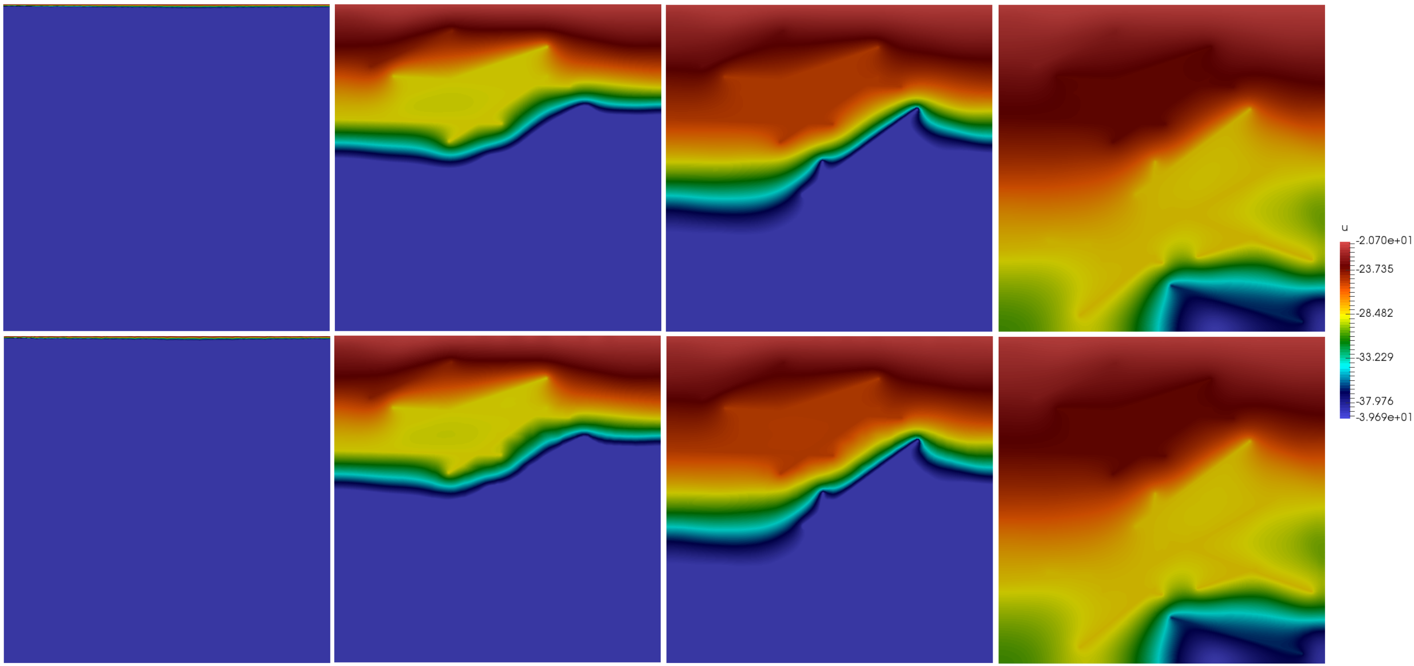
Noew we consider our problem with heterogeneous coefficient k_m with $k_f = 10^9$ for fractures.



Coefficient k_m.

Numerical results

2D heterogeneous fractured medium



Fine scale solution(top line) and multsicale solution using 16 basis functions (bottom line). The results presented on 1, 50, 100, 200 time layers.

Numerical results

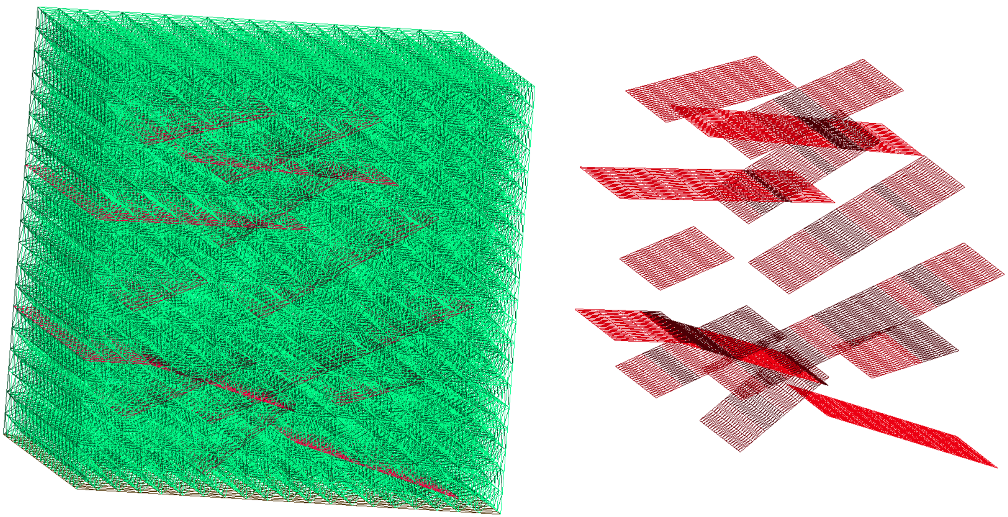
2D fractured heterogeneous medium. L_2 error

Number of bases	DOF_c	t_1	t_{50}	t_{100}	t_{200}
1	121	83.43	92.07	92.07	92.07
2	242	6.98	1.59	4.69	5.05
4	484	2.16	0.37	0.26	0.26
8	968	0.81	0.15	0.08	0.09
12	1452	0.44	0.11	0.05	0.05
16	1936	0.28	0.09	0.06	0.01

L_2 error for different number of basis functions. The size of fine grid system is 14.5 thousands.

Numerical results
3D fractured medium

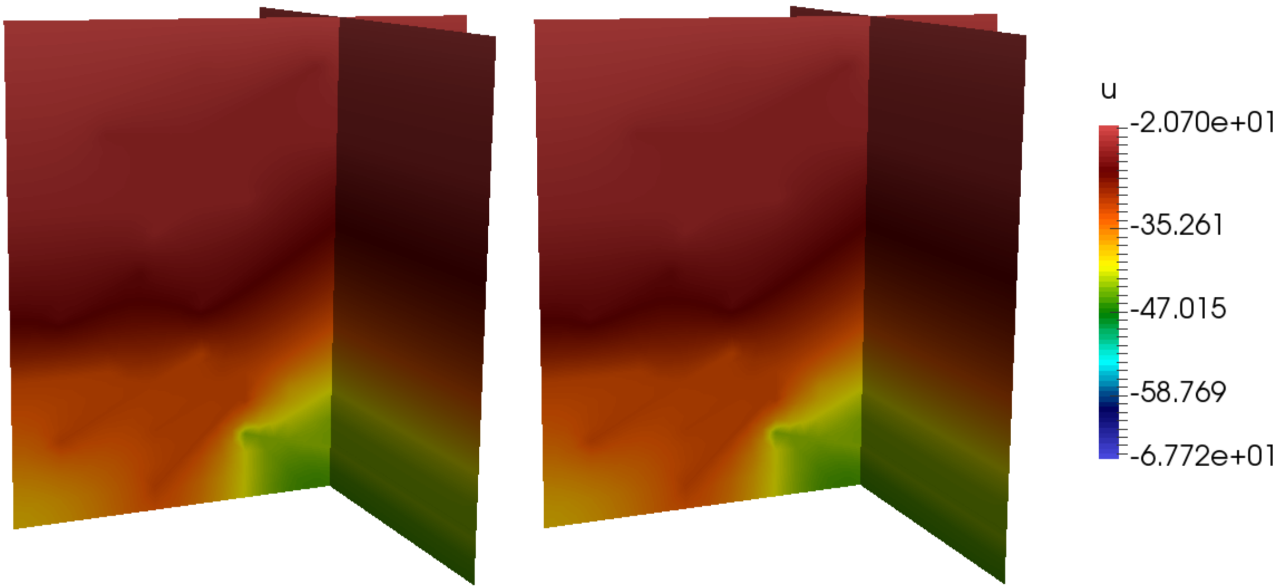
Now we consider 3D problem in fractured domain. As boundary conditions we use $p_1 = -20.7$ on top boundary surface, $p_0 = -61.5$ for initial conditions and $k_m = 10^2, k_f = 10^9$. In this case we took maxinum time equal to 0.0125 s. and 200 time layers.



Computational domain and fine grid with 132,5 thousands cells and 18,6 thousands vertices.

Numerical results

3D fractured medium

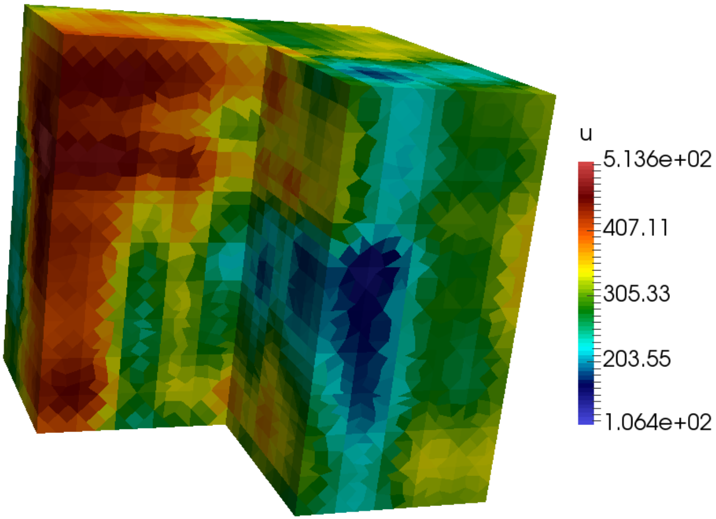


Fine scale solution(top line) and multsicale solution using 16 basis functions (bottom line). L_2 error 1.03%. The results presented on the last time layers.

Numerical results

3D fractured hetrogeneous medium

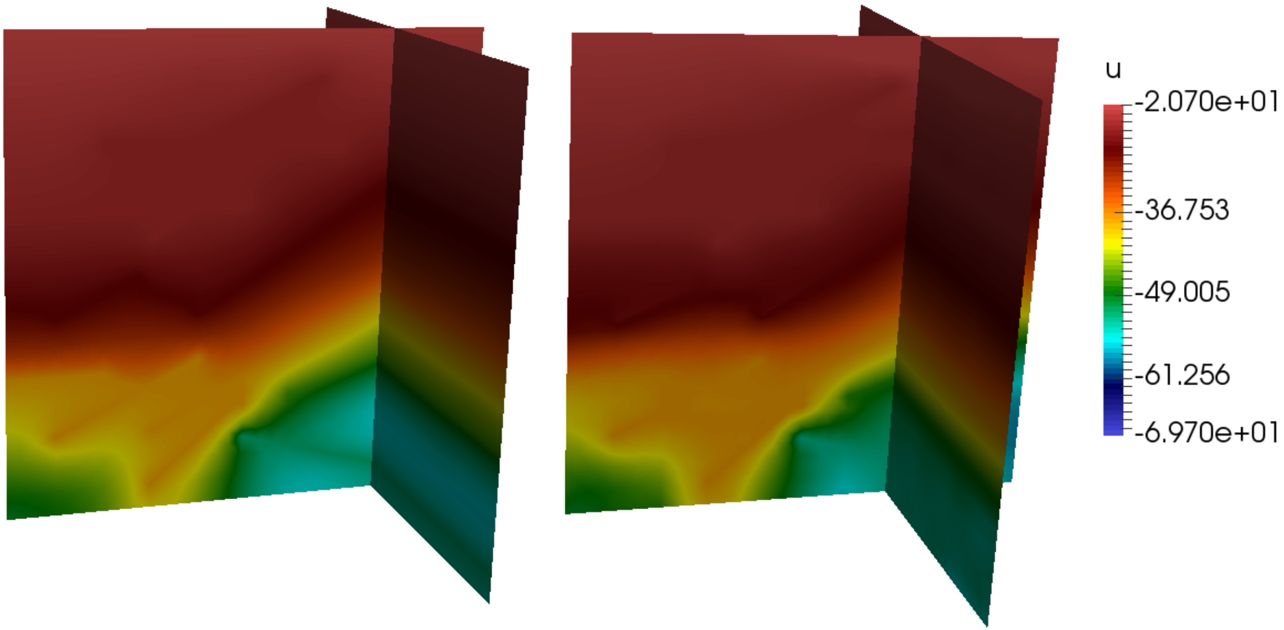
Now we consider 3D problem in fractured heterogeneous domain. We take heterogeneous coefficient k_m and $k_f = 10^9$. In this case we took maxinum time equal to 0.0031 s. and 200 time layers.



Coefficient k_m.

Numerical results

3D fractured heterogeneous medium



Fine scale solution(top line) and multsicale solution using 16 basis functions (bottom line). L_2 error 1.46%. The results presented on the last time layers.

Conclusion

- We described Generalized Multiscale Finite Element method for Richards equation in heterogenous fractured media.
- We presented coarse grid approximation using GMsFEM
- We made comparison of GMsFEM solution with fine-scale solution.
- We presented 2D and 3D results for different number of multiscale basis functions.

Thank you for attention