Approximate the flux variable: a simplified mixed finite element method [*] [†]

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II International conference “Multiscale methods and Large-scale Scientific Computing” Aug. 15–17, 2018
Marchuk Institute of Numerical Mathematics of Russian Academy of Sciences (INM RAS), Gubkin str., 8, Moscow, Russia

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[†]Supported in part by NSF and NRF

Darcy flow model

- Single phase flows in porous media by the Darcy equation:

$$
\begin{aligned}
\mathrm{u} &= -\frac{\mathbb{K}}{\mu}\left(\nabla \mathrm{p} + \rho \mathbb{G}\right), \quad \Omega, && \text{(1a)}\\
\nabla \cdot \mathrm{u} &= \mathrm{f}, \quad \Omega, && \text{(1b)}\\
\boldsymbol{\nu} \cdot \mathrm{u} &= \mathrm{g}, \quad \Gamma. && \text{(1c)}
\end{aligned}
$$

- Ω: a porous medium in $\mathbb{R}^{\mathrm{d}}, \mathrm{d} = 2, 3,$ with Lipschitz boundary $\Gamma = \partial\Omega$.
- u and p: the Darcy flow rate and the pressure in the fluid
- $\mathbb{K}$, μ, and ρ represent the permeability tensor of medium and the viscosity and the density of fluid, and $\mathbb{G}$ denotes the gravity vector,
- $\mathbb{K} \in \mathbb{R}^{\mathrm{d}\times\mathrm{d}}$ is a $\mathrm{d} \times \mathrm{d}$ symmetric and uniformly positive definite matrix and $\mu \in \mathrm{L}^{\infty}(\Omega)$ with $\mathbb{K} > \mathbb{O}$ and $\mu > 0$
- f and g denote the volumetric flow rate source or sink and the prescribed normal flux

Model continued

- Introduce the modified pressure $\widetilde{\mathrm{p}} = \mathrm{p} + \rho \mathrm{g_c z}$ so that
$$\mathrm{u} = -\frac{\mathbb{K}}{\mu}\nabla\widetilde{\mathrm{p}}. \tag{2}$$
- From now on, we will use p in place of $\widetilde{\mathrm{p}}$ and use $\mathscr{A}$ in place of $\frac{\mathbb{K}}{\mu}$.
- Under the assumption that $\mathrm{f} \in \mathrm{H}^{-1}(\Omega)$ and $\mathrm{g} \in \mathrm{H}^{-\frac{1}{2}}(\Gamma)$, with the compatibility condition:
$$\langle \mathrm{f}, 1\rangle_{\mathrm{H}^{-1}(\Omega),\mathrm{H}^{1}(\Omega)} = \langle \mathrm{g}, 1\rangle_{\mathrm{H}^{-1/2}(\Gamma),\mathrm{H}^{1/2}(\Gamma)}, \tag{3}$$
there exists a unique solution $\mathrm{p} \in \mathrm{H}^1(\Omega)/\mathbb{R}$
- For any $\mathrm{g} \in \mathrm{H}^{-\frac{1}{2}}(\Gamma)$, set
$$\mathrm{H_g}(\mathrm{div};\Omega) = \{\mathrm{v} \in \mathrm{H}(\mathrm{div};\Omega) \mid \boldsymbol{\nu}\cdot\mathrm{v} = \mathrm{g} \text{ on } \Gamma\}.$$
- $\mathrm{L}_0^2(\Omega) = \{\mathrm{q} \in \mathrm{L}^2(\Omega) \mid \int_\Omega \mathrm{q}\,\mathrm{dx} = 0\}$.
- The standard mixed weak formulation: find $(\mathrm{u},\mathrm{p}) \in \mathrm{H_g}(\mathrm{div};\Omega) \times \mathrm{L}_0^2(\Omega)$
$$(\mathscr{A}^{-1}\mathrm{u},\mathrm{v}) - (\mathrm{p},\nabla\cdot\mathrm{v}) = 0 \quad \forall \mathrm{v} \in \mathrm{H}_0(\mathrm{div};\Omega), \tag{4a}$$
$$(\nabla\cdot\mathrm{u},\mathrm{q}) = (\mathrm{f},\mathrm{q}) \quad \forall \mathrm{q} \in \mathrm{L}_0^2(\Omega), \tag{4b}$$

Model continued+Mixed FEM

- Choose a suitable $u_g \in H_g(\mathrm{div}; \Omega)$.
- The solution $(u, p) \in H_g(\mathrm{div}; \Omega) \times L_0^2(\Omega)$ is identical to $(w + u_g, p)$ where $(w, p) \in H_0(\mathrm{div}; \Omega) \times L_0^2(\Omega)$ solves

$$
\begin{aligned}
(\mathscr{A}^{-1} w, v) - (p, \nabla \cdot v) &= -(\mathscr{A}^{-1} u_g, v) \quad \forall\, v \in H_0(\mathrm{div}; \Omega), && (5a)\\
(\nabla \cdot w, q) &= (f - \nabla \cdot u_g, q) \quad \forall\, q \in L_0^2(\Omega). && (5b)
\end{aligned}
$$

Mixed Finite Element Method: Let $V_{h,g}^{(k)} \times W_{h,0}^{(k)} \subset H_g(\mathrm{div}; \Omega) \times L_0^2(\Omega)$ be a mixed FEM subspace. Find $(u_h, p_h) \in V_{h,g}^{(k)} \times W_{h,0}^{(k)}$:

$$
\begin{aligned}
(\mathscr{A}^{-1} u_h, v_h) - (p_h, \nabla \cdot v_h) &= 0 \quad \forall v_h \in V_{h,0}^{(k)}, && (6a)\\
(\nabla \cdot u_h, q_h) &= (f, q_h) \quad \forall q_h \in W_{h,0}^{(k)}. && (6b)
\end{aligned}
$$

Mixed Finite Element Spaces $V_{h,g}^{(k)} \times W_{h,0}^{(k)}$

$$\begin{cases} \mathbb{RT}_h^{(k)} := \{v \in H(\mathrm{div};\Omega) \,:\, v|_K \in [P_k(K)]^n \oplus \mathrm{Span}\{x\widetilde{P}_k(K)\},\ K \in \mathscr{T}_h\}, \\ \mathbb{BDM}_h^{(k)} := \{v \in H(\mathrm{div};\Omega) \,:\, v|_K \in [P_k(K)]^n, \quad K \in \mathscr{T}_h\}, \end{cases},$$

- $P_k(K)$ = the space of all polynomials up to degree k defined on K,
  $\widetilde{P}_k(K)$ = the space of all homogeneous polynomials of degree k
- $C^0(P_h^{(k)})$ the standard C^0-conforming FE space of piecewise polynomials of degree $\le k$ on mesh $\mathscr{T}_h$.
- $C^{-1}(P_h^{(k)}) = \{q \in L^2(\Omega) \,:\, q|_K \in P_k(K),\, K \in \mathscr{T}_h\}$,
  $C_0^{-1}(P_h^{(k)}) = C^{-1}(P_h^{(k)})/\mathbb{R} = \{q \in C^{-1}(P_h^{(k)}) \mid \int_\Omega q\, dx = 0\}$.
- $\mathbb{RT}_{h,g}^{(k)} = \mathbb{RT}_h^{(k)} \cap H_g(\mathrm{div};\,\Omega)$, $\quad \mathbb{BDM}_{h,g}^{(k)} = \mathbb{BDM}_h^{(k)} \cap H_g(\mathrm{div};\,\Omega)$,
- The family of RTN/BDM-BDDF Mixed FEM spaces of index k

$$\mathbb{M}_{h,g}^{(k)} := V_{h,g}^{(k)} \times W_{h,0}^{(k)} := \begin{cases} \mathbb{RT}_{h,g}^{\iota(k)} \times C_0^{-1}(P_h^{(k)}), \\ \mathbb{BDM}_{h,g}^{\iota(k)} \times C_0^{-1}(P_h^{(k)}), \end{cases}$$

$$\iota(k) = \begin{cases} k, & \text{if } V_h^{(k)} = \mathbb{RT}_h^{\iota(k)}, \\ k+1, & \text{if } V_h^{(k)} = \mathbb{BDM}_h^{\iota(k)}. \end{cases}$$

Two–step Hybrid Finite Element Method

(with JaEun Ku and Young Ju Lee (ESAIM: Mathematical Modelling and Numerical Analysis 51 (4), 1303-1316, 2017) for homogeneous Dirichlet boundary condition)

Mixed Finite Element Method: Find $(u_h, p_h) \in V_{h,g}^{(k)} \times W_{h,0}^{(k)}$:

$$
\begin{aligned}
(\mathscr{A}^{-1}u_h, v_h) - (p_h, \nabla \cdot v_h) &= 0 \quad \forall v_h \in V_{h,0}^{(k)}, && (7a)\\
(\nabla \cdot u_h, q_h) &= (f, q_h) \quad \forall q_h \in W_{h,0}^{(k)}. && (7b)
\end{aligned}
$$

Step 1 (Coarse-grid solution) On a coarse mesh $\mathscr{T}_H$, obtain the standard Galerkin solution p_H^G satisfying

$$
(\mathscr{A}\nabla p_H^G, \nabla q_H) = (f, q_H) \quad \forall q_H \in C_0^0(P_H^{(k)}). \tag{8}
$$

Step 2 (Fine-grid solution) On a finer mesh $\mathscr{T}_h$, find $u_h \in V_{h,g}^{(k)}$ such that

$$
(\nabla \cdot u_h, \nabla \cdot v_h) + \delta(\mathscr{A}^{-1}u_h, v_h) = (f, \nabla \cdot v_h) + \delta(p_H^G, \nabla \cdot v_h) \quad \forall\, v_h \in V_{h,0}^{(k)}. \tag{9}
$$

Advantages of New Hybrid Method

- Well-developed fast solvers on fine grid (Arnold–Falk–Winther (1997,2000), $\mathrm{H}(\mathrm{div})$ multigrid preconditioner for $\Lambda(\mathrm{u},\mathrm{v}) = \rho^2(\mathrm{u},\mathrm{v}) + \kappa^2(\nabla\cdot\mathrm{u},\nabla\cdot\mathrm{v})$; Hiptmair–Xu (2007, Nodal auxillary subspace precond. ...)
- Of substantially smaller problem size.
- Elimination of the need for artificial stabilization techniques (no *inf-suf* condition.)

$$\inf_{\mathrm{q_h}\in \mathrm{W}_\mathrm{h}^{(k)}} \sup_{\mathrm{v_h}\in \mathrm{V}_\mathrm{h}^{(k)}} \frac{(\nabla\cdot\mathrm{v_h},\mathrm{q_h})}{\|\mathrm{v_h}\|_{\mathrm{H}^1(\Omega)}\|\mathrm{q_h}\|_{\mathrm{L}^2(\Omega)}} = \beta > 0. \tag{10}$$

- A practical and sharp *a posteriori* error estimator
- Play with the parameter δ.

Comparison with the Mixed FE solution

Mixed FEM $(u_h^M, p_h^M) \in V_{h,g}^{(k)} \times W_{h,0}^{(k)}$

$$
\begin{aligned}
(\mathscr{A}^{-1} u_h^M, v_h) - (\nabla \cdot v_h, p_h^M) &= 0 \quad \forall v_h \in V_{h,0}, \\
(\nabla \cdot u_h^M, q_h) &= (f, q_h)
\end{aligned}
$$

$$
\begin{aligned}
\delta(\mathscr{A}^{-1} u_h^M, v_h) - \delta(\nabla \cdot v_h, p_h^M) &= 0, \\
(\nabla \cdot u_h^M, q_h) &= (f, q_h)
\end{aligned}
$$

Taking $q_h = \nabla \cdot v_h$,

$$
(\nabla \cdot u_h^M, \nabla \cdot v_h) + \delta(\mathscr{A}^{-1} u_h^M, v_h) = (f + \delta p_h^M, \nabla \cdot v_h).
$$

Two–step Hybrid Method $(u_h, p_H^G) \in V_{h,g}^{(k)} \tilde{\times} C_0^0(P_H^{(k)})$

$$
(\nabla \cdot u_h, \nabla \cdot v_h) + \delta(\mathscr{A}^{-1} u_h, v_h) = (f + \delta p_H^G, \nabla \cdot v_h) \quad \forall v_h \in V_{h,0}, .
$$

Difference between the Two–step Hybrid Method and the MFEM

$$
(\nabla \cdot (u_h^M - u_h), \nabla \cdot v_h) + \delta(\mathscr{A}^{-1}(u_h^M - u_h), v_h) = \delta(p_h^M - p_H^G, \nabla \cdot v_h).
$$

Basic Error Estimates with H^2 regularity

Theorem (In case of full regularity, i.e., if $u \in H^2$,)

$$
\begin{aligned}
\|u - u_h\|_{L_2(\Omega)} &\le C\|u - \Pi_h u\|_{L_2(\Omega)} \\
&\quad + \sqrt{\delta} h \|u - \Pi_h u\|_{H(\mathrm{div})} + C\sqrt{\delta} H \|p - p_H^G\|_{W_2^1(\Omega)} \\
&\le C h \|u\|_1 + C h^2 \|\nabla \cdot u\|_1 + C\sqrt{\delta} H^2 \|p\|_2 .
\end{aligned}
$$

If $\iota(k) = 0$, then the optimal choice is

$$
h = \sqrt{\delta} H^2 .
$$

New Simple approximation scheme

(submitted with Imbumn Kim, JaEun Ku, and Young Ju Lee)

Scheme (Simple single–step scheme)

Find $u_h \in V_{h,g}^{(k)}$ *such that*

$$(\nabla \cdot u_h, \nabla \cdot v_h) + \delta(\mathscr{A}^{-1} u_h, v_h) = (f, \nabla \cdot v_h) \quad \forall v_h \in V_{h,0}^{(k)}. \tag{11}$$

- Free from the Mixed FE pair of discrete LBB condition.
- Even do not solve for coarse mesh pressure approximation p_H^G
- Not need any discrete function space for pressure if we want to approximate only flux.

Proposition (Quasi-orthogonality property)

$$(\nabla \cdot (u - u_h), \nabla \cdot v_h) + \delta(\mathscr{A}^{-1}(u - u_h), v_h) = \delta(p, \nabla \cdot v_h), \; v_h \in V_{h,0}^{(k)}, \tag{12}$$

Error analysis for the flux

Theorem

Assume that the solution $(\mathrm{u}, \mathrm{p}) \in \mathrm{H}_{\mathrm{g}}(\mathrm{div};\, \Omega) \times \mathrm{L}^2_0(\Omega)$ *to* (4) *belongs to* $\mathrm{H}^{\mathrm{r}-1+\alpha}(\Omega) \times \mathrm{H}^{\mathrm{r}+\alpha}(\Omega)$ *for some* $\mathrm{r} \geq 1$ *and* $\alpha \in [0, 1)$. *For* $\iota(\mathrm{k}) + 1 \leq \mathrm{r}$*, let* $\mathrm{u}_{\mathrm{h}} \in \mathrm{V}^{(\mathrm{k})}_{\mathrm{h},\mathrm{g}}$ *be the solution satisfying* (11). *Then,*

$$\|\mathrm{u} - \mathrm{u}_{\mathrm{h}}\| \leq \mathrm{C}\left(\mathrm{h}^{\mathrm{s}} + \sqrt{\delta}\right)\|\mathrm{u}\|_{\mathrm{s}}, \tag{13}$$

for $0 \leq \mathrm{s} \leq \min(\mathrm{r} + \alpha - 1, \iota(\mathrm{k}) + 1)$. *Furthermore, if* $\nabla \cdot \mathrm{u} \in \mathrm{H}^{\mathrm{r}-1+\alpha}(\Omega)$ *in addition, we have*

$$\|\nabla \cdot (\mathrm{u} - \mathrm{u}_{\mathrm{h}})\| \leq \mathrm{C}(\mathrm{h}^{\mathrm{s}} + \delta)\left(\|\nabla \cdot \mathrm{u}\|_{\mathrm{s}} + \|\mathrm{u}\|_{\mathrm{s}}\right). \tag{14}$$

for $0 \leq \mathrm{s} \leq \min(\mathrm{r} + \alpha - 1, \mathrm{k} + 1)$.

Numerical Example

- $\Omega = (0, 100) \times (0, 20)$
- $\mathrm{p} = 1000$ for $\mathrm{x} = 0$(Left bdry); $\mathrm{p} = 10$ for $\mathrm{x} = 100$(Right bdry); $\boldsymbol{\nu} \cdot \mathrm{u} = 0$ for $\mathrm{y} = 0$ or $\mathrm{y} = 20$(Top & Bottom bdry)
- $\mathrm{h} = 0.5$; $\delta = 2.5 \times 10^{-5}$
- The numerical results agree well with those by the standard MFEM approximation.
- Heterogeneous permeability tensor:

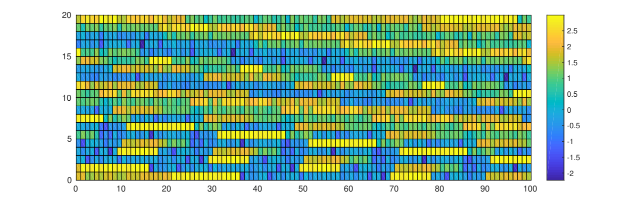

Figure: Permeability tensor with log scale from SPE10 model 1

Numerical Example

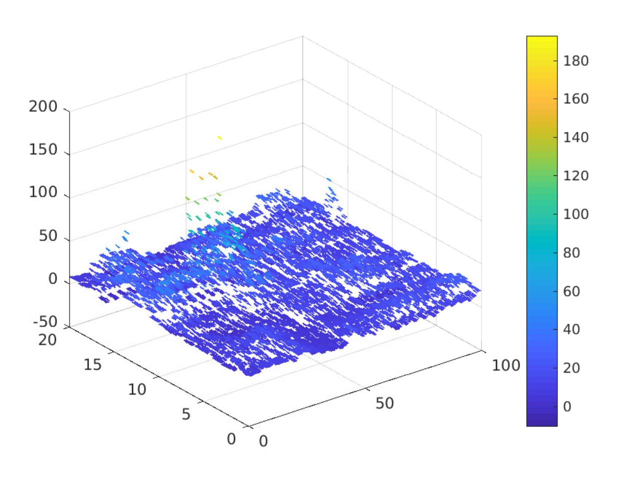

Figure: The x-component of velocity fields by Simple single-step scheme on $\mathbb{RT}^{(0)}_{\mathrm{g,h}}$.

Numerical Example

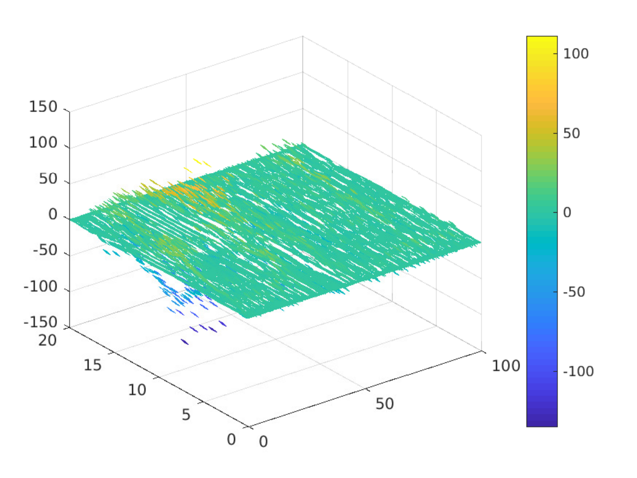

Figure: The y-component of velocity fields by Simple single-step scheme on $\mathbb{RT}^{(0)}_{\mathrm{g,h}}$.

Numerical Example

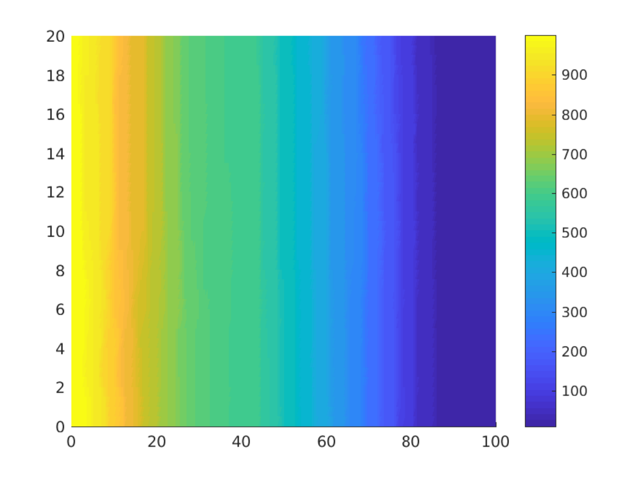

Figure: The pressure p obatined by Simple single-step scheme on $\mathbb{RT}_{\mathrm{g,h}}^{(0)}$ and then on $\mathrm{C}^{-1}(\mathrm{P}_{\mathrm{h}}^{(0)})$.

Numerical Example

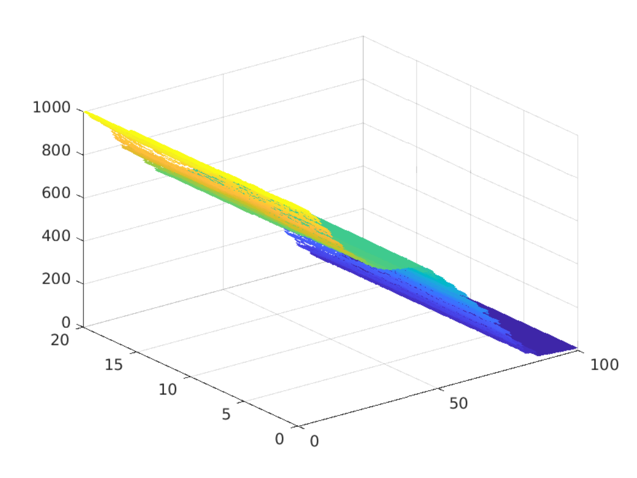

Figure: The pressure p from another direction by Simple single-step scheme on $\mathbb{RT}^{(0)}_{\mathrm{g,h}}$.

Iterative Improvement Approximation to u_h

(In preparation with JaEun Ku)

Scheme (Iterative Improvement: Generation of $(u_h^{(n)})_{n=0}^{\infty} \subset V_{h,g}^{(k)}$)

1. *Choose* $u_h^{(0)} \in V_{h,g}^{(k)}$ *such that* $P_h(\mathscr{A}^{-1}u_h^{(0)}) \in \nabla_h W_{h,\Gamma_\nu}^{(k)}$;
2. *For* $n = 1, 2, \cdots$, *calculate* $u_h^{(n)} \in V_{h,g}^{(k)}$ *fulfilling*

$$(\nabla \cdot u_h^{(n)}, \nabla \cdot v_h) + \delta(\mathscr{A}^{-1}u_h^{(n)}, v_h) = (f, \nabla \cdot v_h) + \delta(\mathscr{A}^{-1}u_h^{(n-1)}, v_h) \quad \forall v_h \in V_{h,0}^{(k)}. \tag{15}$$

Remark (Practic choice of initial approximation)

Choose $u_h^{(0)} \in V_{h,g}^{(k)}$ *such that* $P_h(\mathscr{A}^{-1}u_h^{(0)}) \in V_{h,g}^{(k)}$ *is recommended as follows: find* $w_h \in V_{h,g}^{(k)}$ *fulfilling*

$$(w_h, v_h) = -(f, \nabla \cdot v_h) \quad \forall v_h \in V_{h,0}^{(k)}.$$

Indeed, $w_h = \nabla_h P_h f$. *Then, choose* $u_h^{(0)} = \mathscr{A} w_h$.

Approximation property

The following estimate shows that the new approximation $u_h^{(n)}$ is a better approximation than the original $u_h^{(0)}$.

Proposition

Let $u_h^{(0)} \in V_{h,g}^{(k)}$ *be an approximation to* u *of Step 1, and let* $(u_h^M, p_h^M) \in \mathbb{M}_h^{(k)}$ *be the mixed finite element solution satisfying (7). Let* $u_h^{(n)} \in V_{h,g}^{(k)}$ *be the solution satisfying (15). Then, for* $n \geq 1$, *we have*

$$\|\mathscr{A}^{-1/2}(u_h^{(n)} - u_h^M)\|^2 + \frac{2}{\delta}\|\nabla \cdot (u_h^{(n)} - u_h^M)\| \leq \|\mathscr{A}^{-1/2}(u_h^{(n-1)} - u_h^M)\|^2 \qquad (16)$$

Theorem (Convergence without rate)

$$\lim_{n\to\infty} \|u_h^{(n)} - u_h^M\|^2 = 0.$$

Numerical examples

- Let $\Omega = (0,1)^2$ and consider

$$\begin{aligned} u + \nabla p &= 0, \quad \Omega, && (17a)\\ \nabla \cdot u &= f, \quad \Omega, && (17b)\\ p &= 0, \quad \partial\Omega. && (17c) \end{aligned}$$

- f : generated by the analytic solution $p(x,y) = (x - x^2)(y - y^2)$.
- Uniform triangulations of Ω are adopted in our numerical simulation;
- $\mathbb{RT}^{(0)}_{h,\Gamma_\nu}$ space is employed;
- Initial guess $u_h^{(0)} = \vec{0}$
- $\|u_h^{(n)} - u_h^{(n-1)}\| < 10^{-10}$.

Numerical results $\delta = \frac{1}{2^6}$

δ=1; $h = \frac{1}{2^6}$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$\lVert u - u_h^{(n)} \rVert$	7.49E-3	2.35E-3	2.33E-3	2.33E-3	2.33E-3
$\lVert u_h^{(n)} - u_h^{(n-1)} \rVert$	2.02E-2	4.61E-5	1.09E-7	2.49E-10	5.79E-13

δ=0.1; $h = \frac{1}{2^6}$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$\lVert u - u_h^{(n)} \rVert$	2.44E-3	2.33E-3	2.33E-3	-	-
$\lVert u_h^{(n)} - u_h^{(n-1)} \rVert$	2.02E-2	5.50E-7	1.40E-11	-	-

δ=1; $h = \frac{1}{2^7}$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$\lVert u - u_h^{(n)} \rVert$	7.22E-3	1.21E-3	1.16E-3	1.16E-3	1.16E-3
$\lVert u_h^{(n)} - u_h^{(n-1)} \rVert$	2.02E-2	4.61E-5	1.07E-7	2.49E-10	5.79E-13

δ=0.1; $h = \frac{1}{2^7}$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$\lVert u - u_h^{(n)} \rVert$	1.38E-3	1.16E-3	1.16E-3	-	-
$\lVert u_h^{(n)} - u_h^{(n-1)} \rVert$	2.02E-2	5.50E-7	1.40E-11	-	-

Thank you for your attention !!!