Variational stabilization of convection-dominated diffusion with optimal test functions

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Motivation

Numerical methods achieve stability in many different ways

- Standard finite element method — coercivity & conformity
- Mixed methods — balanced pair of spaces
- SUPG methods — artificially added streamline diffusion
- DG methods — upwind stabilization & jump penalization
- HDG methods — difference between interior & interface unknowns

  ⋮ ⋮
- DPG methods — stability by automatic test space design

Key Difficulty: Exact inf-sup condition $\not\Rightarrow$ Discrete inf-sup condition

Ritz-Galerkin Method

Best Approximation Property (Projection Principle)

- If *a* symmetric, real-valued bilinear form & *elliptic* on Hilbert space U

$$a(\mathbf{v}, \mathbf{v}) \geq \gamma \, ||\mathbf{v}||_U^2 \quad \forall \mathbf{v} \in U$$

it defines an inner product and a norm in U:

$$|||\mathbf{v}||| = \sqrt{a(\mathbf{v}, \mathbf{v})} \quad \forall \mathbf{v} \in U$$

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- Then, the Ritz-Galerkin approximation $\mathbf{u}_h$ of $\mathbf{u} \in U$ in a finite-dimensional subspace $U_h \subset U$ satisfies

$$a(\mathbf{u}_h, \mathbf{v}_h) = a(\mathbf{u}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in U_h$$

which is the *orthogonal projection* of $\mathbf{u}$ onto U_h:

$$a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in U_h$$

- Thus, it is the *best approximation* of $\mathbf{u}$ in U_h:

$$|||\mathbf{u} - \mathbf{u}_h||| = \min_{\mathbf{v}_h \in U_h} |||\mathbf{u} - \mathbf{v}_h|||$$

“Petrov-Galerkin” schemes (PG)

PG schemes define different trial and test (Hilbert) spaces

Problem: $\left[\begin{array}{l}\text{PDE}+\\ \text{boundary conditions}\end{array}\right.$

$\downarrow$

Variational form: $\left[\begin{array}{c}\text{Find } u \text{ in a trial space } U \text{ satisfying}\\ b(u,w)=\ell(w)\\ \text{for all } w \text{ in a test space } W\end{array}\right.$

$\downarrow$

Discretization: $\left[\begin{array}{c}\text{Find } u_h \text{ in a discrete trial space } U_h \subset U \text{ satisfying}\\ b(u_h,w_h)=\ell(w_h)\\ \text{for all } w_h \text{ in a discrete test space } W_h \subset W\end{array}\right.$

For PG schemes, $U_h \neq W_h$ in general

Optimal Petrov-Galerkin Method

- Consider a general variational problem

$$\text{Find } \mathbf{u} \in U \quad \text{s.t.} \quad b(\mathbf{u}, \mathbf{w}) = l(\mathbf{w}) \quad \forall \mathbf{w} \in W$$

and let $U_h \subset U$ be a finite-dimensional trial subspace

$$U_h = \operatorname{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$$

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  and let $U_h \subset U$ be a finite-dimensional trial subspace

$$U_h = \text{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$$

- The corresponding *optimal test space* is defined as

$$W_h^{\text{opt}} = \text{span}\{\mathbf{T}\mathbf{e}_1, \ldots, \mathbf{T}\mathbf{e}_N\} \subset W$$

  where the trial-to-test map $\mathbf{T} : U \to W$ is defined through

$$(\mathbf{T}\mathbf{u}, \delta\mathbf{w})_W = b(\mathbf{u}, \delta\mathbf{w}) \quad \forall \delta\mathbf{w} \in W$$

- Let each $\mathbf{w}_h \in W_h^{\text{opt}}$ be $\mathbf{w}_h = \mathbf{T}\mathbf{v}_h$ for some $\mathbf{v}_h \in U_h$, then

$$b(\mathbf{u}_h, \mathbf{w}_h) = b(\mathbf{u}_h, \mathbf{T}\mathbf{w}_h) = (\mathbf{T}\mathbf{u}_h, \mathbf{T}\mathbf{w}_h)_W \stackrel{\text{def}}{=} a(\mathbf{u}_h, \mathbf{v}_h)$$

$$l(\mathbf{w}_h) = l(\mathbf{T}\mathbf{v}_h) \stackrel{\text{def}}{=} Q(\mathbf{v}_h)$$

Optimal Petrov-Galerkin Method (cont.)

- Optimal PG delivers best approximation in generalized energy norms:

$$|||\mathbf{u}|||^2 = a(\mathbf{u},\mathbf{u}) = (\mathbf{T}\mathbf{u},\mathbf{T}\mathbf{u})_W = ||\mathbf{T}\mathbf{u}||_W^2 = \left\{ \sup_{\mathbf{w}\in W} \frac{|b(\mathbf{u},\mathbf{w})|}{||\mathbf{w}||_W} \right\}^2$$

- Ellipticity of $a(\mathbf{u},\mathbf{u})$ induces the inf-sup condition on $b(\mathbf{u},\mathbf{v})$

Optimal Petrov-Galerkin Method (cont.)

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- Ellipticity of $a(\mathbf{u}, \mathbf{u})$ induces the inf-sup condition on $b(\mathbf{u}, \mathbf{v})$
- Q1: How to determine optimal test function space in a practice?
- A1: Computed (almost) *automatically* within DPG framework!

Optimal Petrov-Galerkin Method (cont.)

- Optimal PG delivers best approximation in generalized energy norms:

$$|||\mathbf{u}|||^2 = a(\mathbf{u},\mathbf{u}) = (\mathbf{Tu},\mathbf{Tu})_W = ||\mathbf{Tu}||_W^2 = \left\{ \sup_{\mathbf{w}\in W} \frac{|b(\mathbf{u},\mathbf{w})|}{||\mathbf{w}||_W} \right\}^2$$

- Ellipticity of $a(\mathbf{u},\mathbf{u})$ induces the inf-sup condition on $b(\mathbf{u},\mathbf{v})$
- Q1: How to determine optimal test function space in a practice?
- A1: Computed (almost) *automatically* within DPG framework!
- Q2: What if $|||\cdot|||$ has harmful parameter dependencies?
- A2: We can have $|||\cdot||| = ||\cdot||_U$ if we select

$$||\mathbf{w}||_W = |||\mathbf{w}|||_{W,\mathrm{opt}} = \sup_{\mathbf{u}\in U} \frac{|b(\mathbf{u},\mathbf{w})|}{||\mathbf{u}||_U}$$

Motivation

Stabilized FEMs resolve the numerical instability issue.

1. Least squares FEM (LSFEM)
2. Galerkin method with least squares (G/LS)
3. Streamlined-upwind Petrov-Galerkin (SUPG) method
4. Variational multi-scale (VMS) method
5. Discontinuous Petrov-Galerkin method (DPG)

1 – Overly diffuse solutions on coarse meshes, limits $f \in L^2(\Omega)$
2-4 – Requires fine-tuning of penalty/stabilization parameters
5 – Introduces additional degrees of freedom (DOFs)

Target Problem

Convection Dominated Diffusion Problems

Find u such that:

$$-\nabla \cdot \boldsymbol{\kappa} \nabla u + \mathbf{a} \cdot \nabla u = f, \text{ in } \Omega \subset \mathbb{R}^{2,3}$$

Boundary conditions:

$$u = 0, \text{ on } \Gamma_D,$$

$$\boldsymbol{\kappa} \nabla u \cdot \mathbf{n} = g, \text{ on } \Gamma_N.$$

- $\boldsymbol{\xi} \cdot \boldsymbol{\kappa} \boldsymbol{\xi} > 0, \quad \forall \boldsymbol{\xi} \neq 0$
- $\|\boldsymbol{\kappa}\|_{L^\infty(\Omega)} \ll \|\mathbf{a}\|_{L^\infty(\Omega)}$

Proposed Approach

- Derive integral statement on broken Hilbert spaces:
  - Test functions in $L^2(\Omega)$ with local higher regularity
    $\Rightarrow$ Reduced local regularity requirements on f
- Compute 'optimal' discontinuous test functions that automatically deliver discrete stability
- Reduce DOF number in discrete approximation
  - Apply Petrov-Galerkin framework
  - Employ classical $C^0(\Omega)$ trial/solution basis functions
  - Incorporate piecewise discontinuous test functions
- Use first order system description (mixed form)
  $\Rightarrow$ enforces normal flux continuity in heterogeneous media

Derivation of the Weak Statement

Step 1: Rewrite BVP in mixed form

Find $(u, \boldsymbol{\sigma})$ such that:

$$\boldsymbol{\kappa}\,\boldsymbol{\nabla} u - \boldsymbol{\sigma} = \mathbf{0}, \text{ in } \Omega$$

$$-\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \mathbf{a} \cdot \boldsymbol{\nabla} u = f, \text{ in } \Omega$$

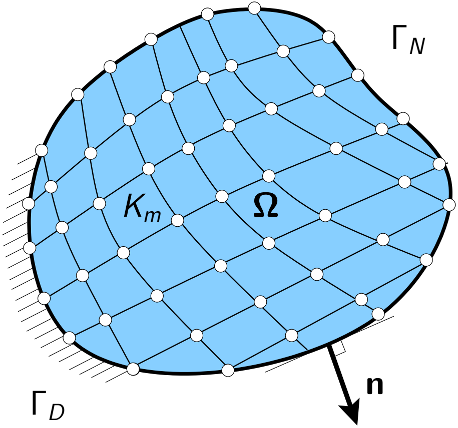
Boundary conditions:

$$u = 0, \text{ on } \Gamma_D$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = g, \text{ on } \Gamma_N$$

Derivation of the Weak Statement

Domain Partition $\mathcal{P}_h = \{K_m\}_{m=1}^{N_{\text{elm}}}$



Derivation of the Weak Statement

Step 2: On each element K_m, enforce PDE weakly, i.e.,

$$\boxed{\begin{array}{l}\text{Find } (u, \boldsymbol{\sigma}) \text{ such that,} \quad \forall (v, \mathbf{w}) \in L^2(K_m) \times [L^2(K_m)]^2 \\[1em] \displaystyle\int_{K_m} \Big\{ [\boldsymbol{\kappa} \nabla u - \boldsymbol{\sigma}] \cdot \mathbf{w} - [\nabla \cdot \boldsymbol{\sigma} + (\mathbf{a} \cdot \nabla u)] v \Big\} d\mathbf{x} = \int_{K_m} \{ f \ v \} d\mathbf{x} \end{array}}$$

Derivation of the Weak Statement

Step 3: Apply Green's Identity

Find $(u, \boldsymbol{\sigma})$ such that, $\quad \forall (v, \mathbf{w}) \in H^1(K_m) \times [L^2(K_m)]^2$

$$\int_{K_m} \left\{ (\boldsymbol{\kappa} \boldsymbol{\nabla} u - \boldsymbol{\sigma}) \cdot \mathbf{w} + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} v - (\mathbf{a} \cdot \boldsymbol{\nabla} v) u \right\} d\mathbf{x}$$
$$+ \oint_{\partial K_m} \left\{ (\mathbf{a} \cdot \mathbf{n})\, u\, v - (\boldsymbol{\sigma} \cdot \boldsymbol{n}) v \right\} ds = \int_{K_m} \{ f\ v \} d\mathbf{x}$$

Derivation of the Weak Statement

Step 4: Apply boundary & continuity conditions, sum local statements:

$$\boxed{\begin{array}{c}\text{Find } (u,\boldsymbol{\sigma}) \in U(\mathcal{P}_h) \text{ such that:}\\[1em] B((u,\boldsymbol{\sigma}),(v,\mathbf{w})) = F(v), \quad \forall (v,\mathbf{w}) \in V(\mathcal{P}_h)\end{array}}$$

where

$$B((u,\boldsymbol{\sigma}),(v,\mathbf{w})) = \sum_{K_m \in \mathcal{P}_h} \Bigg[\int_{K_m} \Big\{ (\boldsymbol{\kappa} \boldsymbol{\nabla} u - \boldsymbol{\sigma}) \cdot \mathbf{w} + (\boldsymbol{\sigma} - u\,\mathbf{a}) \cdot \boldsymbol{\nabla} v \Big\} d\mathbf{x}$$

$$- \oint_{\partial K_m \setminus \partial \Omega} \{ (\mathbf{a} \cdot \mathbf{n})\, u_{\text{neigh}}\, v - (\boldsymbol{\sigma}_{\text{neigh}} \cdot \boldsymbol{n}) v \} ds \Bigg]$$

$$F(v) = \sum_{K_m \in \mathcal{P}_h} \left[\int_{K_m} \{ f\, v \} d\mathbf{x} + \oint_{\partial K_m \cap \Gamma_N} \{ g\, v \} ds \right]$$

Derivation of the Weak Statement

Discrete Spaces

$$U(\mathcal{P}_h) \stackrel{\text{def}}{=} \Big\{(u, \boldsymbol{\sigma}) \in H^1(\mathcal{P}_h) \times [L^2(\Omega)]^2 : \ \boldsymbol{\sigma}_{|_{K_m}} \in H(\text{div}, K_m) \\ \wedge \ \gamma^0(u_{|_{K_m}})_{|_{\partial K_m \cap \Gamma_D}} = 0, \ \forall K_m \in \mathcal{P}_h\Big\}$$

$$V(\mathcal{P}_h) \stackrel{\text{def}}{=} \Big\{(v, \mathbf{w}) \in H^1(\mathcal{P}_h) \times [L^2(\Omega)]^2 : \\ \gamma^0(v_{|_{K_m}})_{|_{\partial K_m \cap \Gamma_D}} = 0, \ \forall K_m \in \mathcal{P}_h\Big\}$$

where:

$$H^1(\mathcal{P}_h) \stackrel{\text{def}}{=} \Big\{v \in L^2(\Omega) : \quad v_m \in H^1(K_m), \ \forall K_m \in \mathcal{P}_h\Big\}$$

FE Discretization: Trial Space Proposition

Find $(u_h, \boldsymbol{\sigma}_h) \in U^h(\mathcal{P}_h)$ such that:

$$B((u_h, \boldsymbol{\sigma}_h), (v_h, \mathbf{w}_h)) = F(v_h), \quad \forall (v_h, \mathbf{w}_h) \in V^*(\mathcal{P}_h)$$

Trial space $U^h(\mathcal{P}_h) \subset U(\mathcal{P}_h)$ consists of piecewise continuous polynomials of degree p, that is:

$$u_h(\mathbf{x}) = \sum_{i=1}^{N} u_i^h e_i(\mathbf{x}), \quad \boldsymbol{\sigma}_h(\mathbf{x}) = \begin{Bmatrix} q_x^h(\mathbf{x}) \\ \\ q_y^h(\mathbf{x}) \end{Bmatrix} = \sum_{j=1}^{N} \begin{Bmatrix} q_{x_j}^h \varepsilon_{x_j}(\mathbf{x}) \\ \\ q_{y_j}^h \varepsilon_{y_j}(\mathbf{x}) \end{Bmatrix}$$

FE Discretization: Test Space Construction

For each trial function $e_i, \varepsilon_{x_j}, \varepsilon_{y_k}$, compute 'optimal' test functions $(v_i, \mathbf{w}_i)$, $(v_j, \mathbf{w}_j)$, & $(v_k, \mathbf{w}_k)$, resp.:

$$\begin{aligned}
((p,\mathbf{r}),(v_i,\mathbf{w}_i))_{V(\mathcal{P}_h)} &= B((e_i,\mathbf{0}),(p,\mathbf{r})), && \forall (p,\mathbf{r})\\
((p,\mathbf{r}),(v_j,\mathbf{w}_j))_{V(\mathcal{P}_h)} &= B((0,(\varepsilon_{x_j},0)),(p,\mathbf{r})), && \forall (p,\mathbf{r})\\
((p,\mathbf{r}),(v_k,\mathbf{w}_k))_{V(\mathcal{P}_h)} &= B((0,(0,\varepsilon_{y_k})),(p,\mathbf{r})), && \forall (p,\mathbf{r})
\end{aligned}$$

LHS is **inner product** on $V(\mathcal{P}_h) \times V(\mathcal{P}_h)$, i.e.,

$$((v,\mathbf{w}),(p,\mathbf{r}))_{V(\mathcal{P}_h)} \stackrel{\text{def}}{=} \sum_{K_m \in \mathcal{P}_h} \left\{ \int_{K_m} \left[h_m^2 \nabla v \nabla p + v\, p + \mathbf{w} \cdot \mathbf{r} \right] \, \mathrm{d}\mathbf{x} \right\}$$

FE Discretization

Remarks:

- Integral statements governing optimal test functions are infinite-dimensional problems $\Rightarrow$ test functions computed numerically
- C^0 trial functions imply that support of test functions is identical to trial functions and solved locally, i.e., element-by-element solution
- Optimal test functions span subspace $V^*(\mathcal{P}_h) \subset V(\mathcal{P}_h)$ $\Rightarrow$ used in FE computation of $(u_h, \boldsymbol{\sigma}_h)$.
- (DPG argument) Optimal test functions imply discrete stability

$$B_{ij} = B((e_i, \mathbf{0}), (v_j, \mathbf{w}_j)) = ((v_j, \mathbf{w}_j), (v_i, \mathbf{w}_i))_{V(\mathcal{P}_h)}.$$

FE Discretization: Summary

Continuous-discontinuous Petrov-Galerkin (cDPG) method

- Introduction of a hybrid FE method:
  - [Petrov-Galerkin Framework]
    - Continuous trial functions (Classical C^0 piecewise polynomials)
    - Discontinuous test functions
  - [DPG] Optimal test functions $\Rightarrow$ unconditional discrete stability
  - [Mixed FEMs/LSFEM] First order systems

- Discrete stability guaranteed without calibrating coefficients

- Comparison to LSFEM
  - Weaker regularity required on source and Neumann data
  - Sufficient for f to be in the dual of $H^1(\mathcal{P}_h)$ and $g \in H^{-1/2}(\Gamma_N)$

Preliminary Numerical Results

- Test functions may use the same local polynomial degree of approximation as trial/solution functions
- Optimal h-convergence rates for $\|u\|_{H^1(\Omega)}$, $\|u\|_{L^2(\Omega)}$ &$\|\boldsymbol{\sigma}\|_{L^2(\Omega)}$
- Comparison alternative methods
  - Convection-dominated diffusion problem
  - 'Shock' problem

Comparison Study - cDPG vs. Other Methods

Model problem: convection diffusion problem on the unit square:

Find u such that:

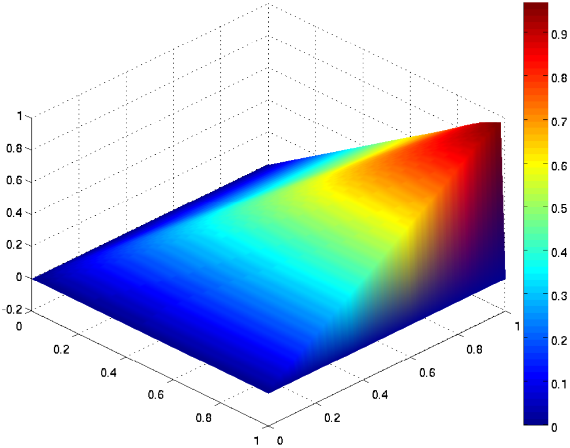
$$\begin{aligned} -\frac{1}{Pe}\Delta u + \mathbf{a}\cdot\boldsymbol{\nabla} u &= 1 \;\text{ in }\; \Omega = (0,1)\times(0,1), \\ u &= 0, \;\text{ on }\; \partial\Omega, \end{aligned}$$

Focus on **convection dominated** scenario: $Pe = 10^6$ & $\mathbf{a} = \{1,1\}^T$

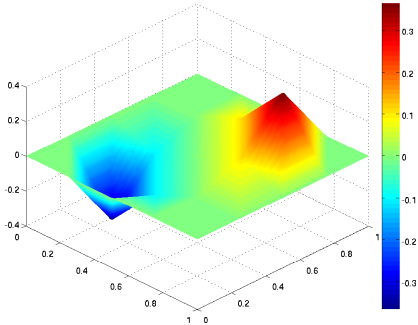
Thus, convection with $\pi/4$ angle, boundary layers along top/right edges (width $\sim \frac{1}{Pe}$)

Convection Dominated Diffusion Problem

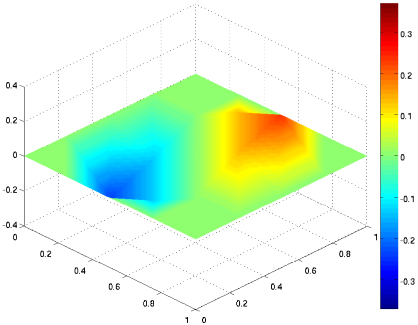
Reference Solution:



Uniform 4 × 4 Coarse Mesh

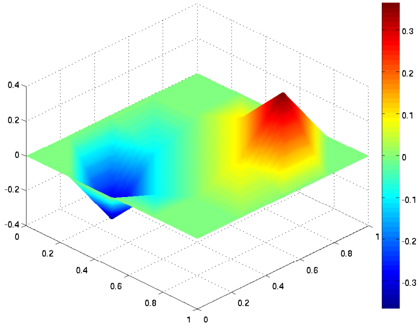


cDPG

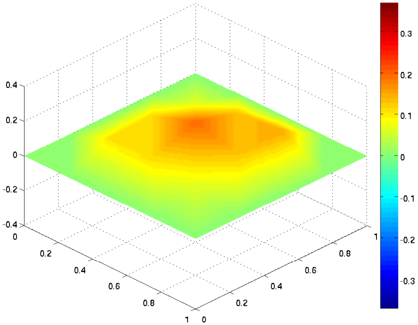


LSFEM

Uniform 4 × 4 Coarse Mesh



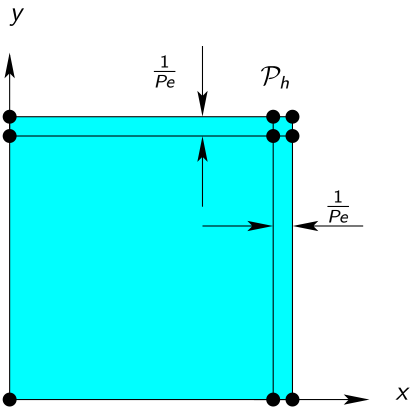
cDPG



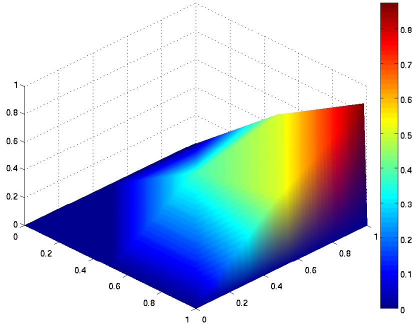
SUPG, VMS, G/LS

Graded Mesh

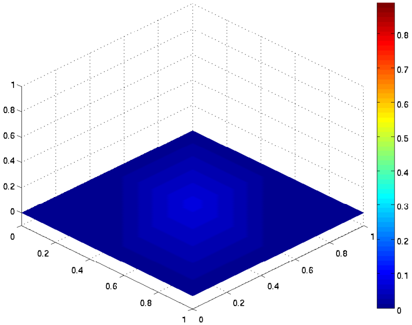
Numerical results for $Pe = 10^6$, starting with graded four-element mesh, $\Omega = (0,1) \times (0,1)$, $p = 2$, with subsequent uniform refinements



4 × 4 Graded Mesh

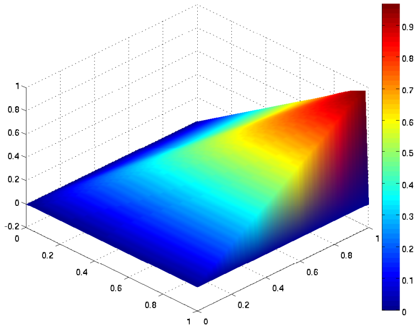


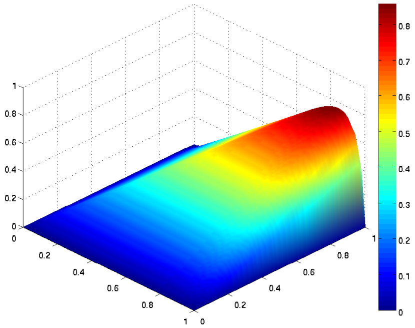
cDPG



SUPG, VMS, and G/LS

Refined Graded Mesh



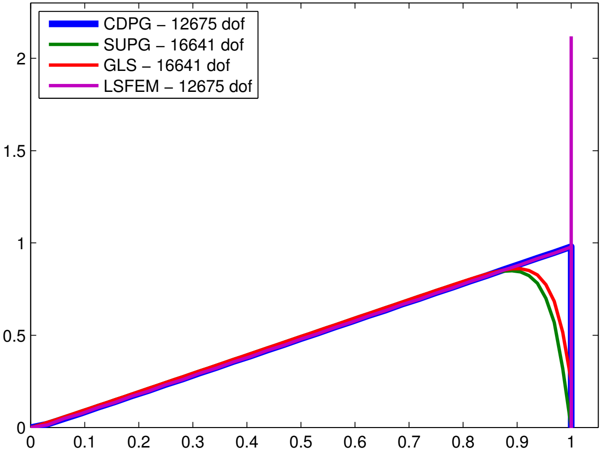


cDPG, LSFEM (12k dofs)

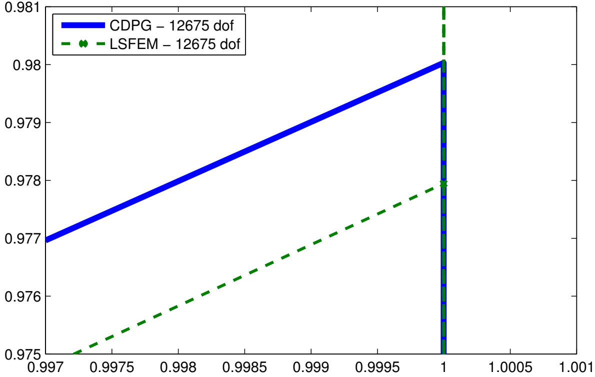
SUPG, **G/LS** (16k dofs)

VMS produced highly oscillatory solutions

Distribution of u Along the Diagonal



(Zoomed in) Distribution of u at the Corner



Comparison Study - 'Shock' Problem

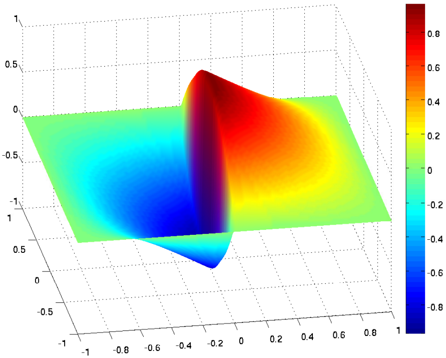
Convection-diffusion problem on the square $\Omega = (-1,1) \times (-1,1)$:

Find u such that:

$$\begin{aligned} -\frac{1}{Pe}\Delta u - \begin{Bmatrix} x \\ 0 \end{Bmatrix} \cdot \boldsymbol{\nabla} u &= x\left(1-y^2\right)\tfrac{2x}{Pe}, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned}$$

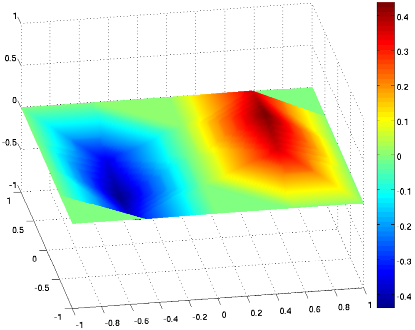
Comparison Study - 'Shock' Problem

Reference Solution:

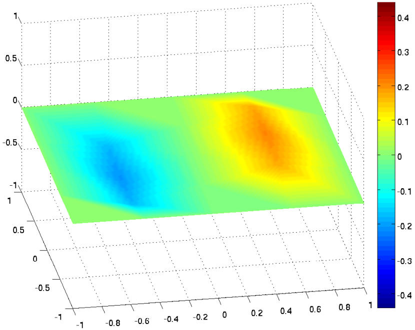


Numerical results for $Pe = 10^9$, uniform meshes, $p = 1$ for cDPG and LSFEM, and $p = 2$ for SUPG, VMS, G/LS.

16×16 Uniform Coarse Mesh

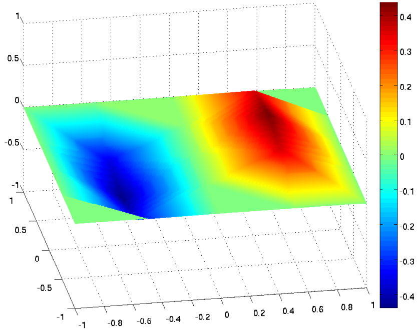


cDPG
$(p = 1)$

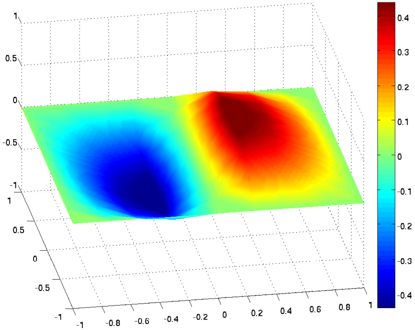


LSFEM
$(p = 1)$

16×16 Uniform Coarse Mesh

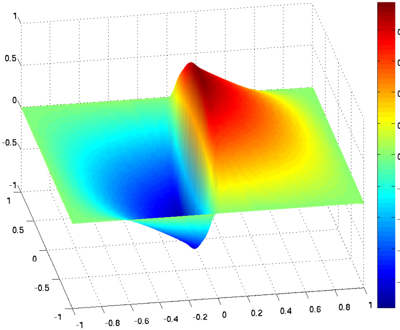


cDPG
$(p = 1)$

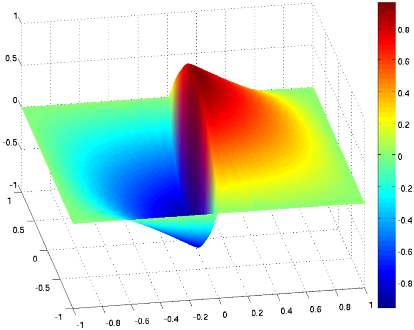


SUPG, VMS, **G/LS**
$(p = 2)$

Uniform Refined Mesh

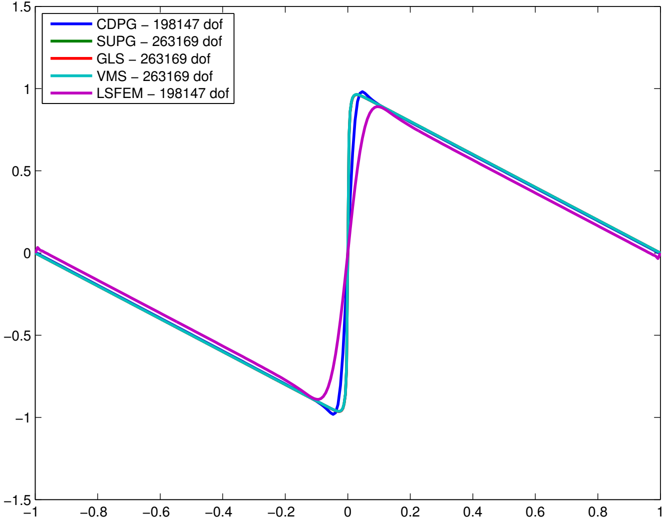


cDPG
($p = 1$, 198k dofs)



SUPG, **G/LS**
($p = 2$, 263k dofs)

Distribution of u Along the Centerline $x = 0$



Preliminary Results - Heterogeneous Domain

Convection-diffusion problem

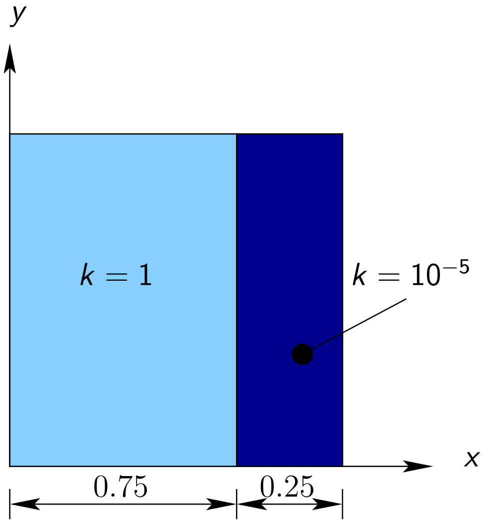
Find u such that:

$$-\boldsymbol{\nabla} \cdot (k(\mathbf{x})\mathbf{I})\boldsymbol{\nabla} u + \mathbf{a} \cdot \boldsymbol{\nabla} u = f, \text{ in } \Omega = (0,1) \times (0,1),$$

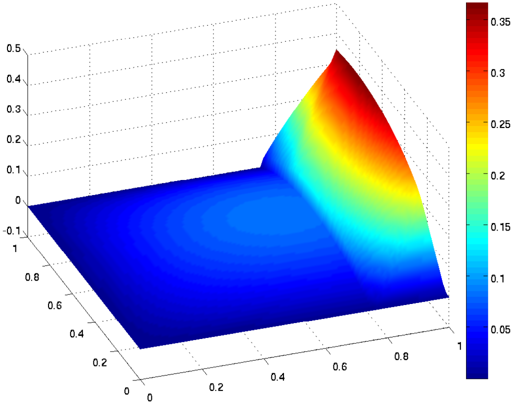
$$u = 0, \text{ on } \partial\Omega,$$

- $\mathbf{a} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$, $f = 1$, and $\boldsymbol{\kappa} = k\boldsymbol{I}$
- Solution space built on uniform meshes with polynomial order $p = 2$
- Test functions solved using $p + 1 = 3$ on the same mesh

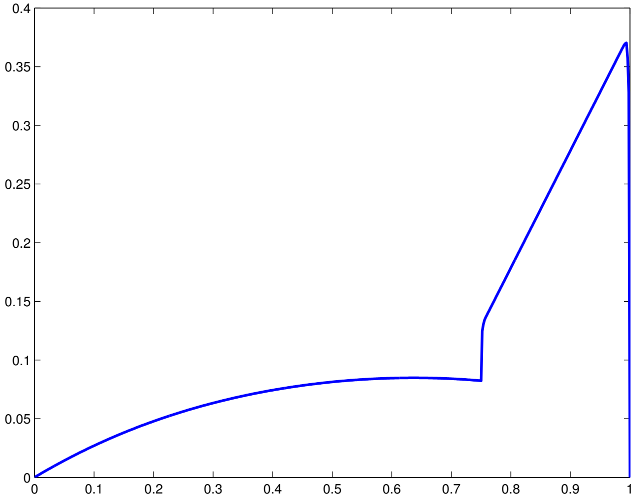
Preliminary Results - Heterogeneous Domain



Preliminary Results - Heterogeneous Domain



Distribution of u along $y = \frac{1}{2}$



Concluding Remarks

- Introduced ybrid continuous-discontinuous Petrov-Galerkin method.
  - Solution/trial functions are piecewise continuous
  - Weight/test functions are piecewise discontinuous.
- DPG approach: test functions computed automatically to establish numerically stable FE approximations
- Support of each discontinuous test function is identical to its corresponding continuous trial function
- Local test-function contribution computed locally (i.e. decoupled)
  - Sufficient accuracy by using corresponding p-level of trial function

Concluding Remarks

- Formulation allows for lower regularity of f (in dual of $H^1(\mathcal{P}_h)$) compared to LSFEM ($f \in L^2(\Omega)$)
- Numerical solutions do not show overly diffuse LSFEM solutions
- Numerical results compete with (SUPG, VMS, G/LS)
  - Show no oscillations at boundary layers.

Remark:

Weaker trial functions for $\boldsymbol{\sigma}$ than C^0, i.e., use trial functions with continuous normal fluxes discontinuous tangential fluxes across element boundaries ($\sim$ Raviart-Thomas/Brezzi-Douglas-Marini approach) deliver similar results to those reported herein