A Finite Element Method For PDEs in Time-Dependent Domains

Maxim A. Olshanskii

University of Houston

www.math.uh.edu/ ~molshan

Collaborator:

Christoph Lehrenfeld (University of Göttingen)

Model problem

Let $\Omega(t) \subset \mathbb{R}^d,\ d = 2, 3$ bounded regular for each $t \in [0, T]$, $T > 0$ and evolves smoothly: $\exists$ one-to-one continuous mapping

$$\Psi(t) \,:\, \Omega_0 \to \Omega(t) \quad \text{for } t \in [0, T].$$

Model problem

Let $\Omega(t) \subset \mathbb{R}^d,\ d = 2, 3$ bounded regular for each $t \in [0, T]$, $T > 0$ and evolves smoothly: $\exists$ one-to-one continuous mapping

$$\Psi(t) : \Omega_0 \to \Omega(t) \quad \text{for } t \in [0, T].$$

One is interested in solving

$$\frac{\partial u}{\partial t} + \mathcal{L}(t, x)u = 0 \quad \text{on} \quad \Omega(t), \quad t \in (0, T],$$
$$+b.c., \quad + i.c.$$

where $\mathcal{L}(t, x)$ is a second order differential operator uniformly elliptic in time.

Model problem

Let $\Omega(t) \subset \mathbb{R}^d,\ d = 2, 3$ bounded regular for each $t \in [0, T]$, $T > 0$ and evolves smoothly: $\exists$ one-to-one continuous mapping

$$\Psi(t) \,:\, \Omega_0 \to \Omega(t) \quad \text{for } t \in [0, T].$$

One is interested in solving

$$\frac{\partial u}{\partial t} + \mathcal{L}(t, x)u = 0 \quad \text{on} \;\; \Omega(t), \quad t \in (0, T], \\ +b.c., \quad +i.c.$$

where $\mathcal{L}(t, x)$ is a second order differential operator uniformly elliptic in time.

Well-posedness analysis: Savare et al (1996, 1997), Prokert (1999), Bonaccorsi & Guatteri (2001) ... Alphonse, Elliott & Stinner (2015)

Model problem

Let $\Omega(t) \subset \mathbb{R}^d,\ d = 2,3$ bounded regular for each $t \in [0,T]$, $T > 0$ and evolves smoothly: $\exists$ one-to-one continuous mapping

$$\Psi(t) \,:\, \Omega_0 \to \Omega(t) \quad \text{for } t \in [0,T].$$

One is interested in solving

$$\frac{\partial u}{\partial t} + \mathcal{L}(t,x)u = 0 \quad \text{on} \;\; \Omega(t), \quad t \in (0,T],$$
$$+b.c., \quad + i.c.$$

where $\mathcal{L}(t,x)$ is a second order differential operator uniformly elliptic in time.

For the analysis, we need $\partial\Omega_0 \in C^{1,1}$ and $\Psi \in C^{r+1}([0,T] \times \overline{\Omega_0})$, where $r \geq 1$ is FE degree.

Model problem (an example)

Consider a smooth motion and deformation of the material volume $\Omega(t)$, e.g., volume of fluid. For $y \in \Omega_0$, Lagrangian mapping $\Psi(t,y)$ solves

$$\Psi(0,y) = y, \quad \frac{\partial \Psi(t,y)}{\partial t} = \mathbf{w}(t, \Psi(t,y)), \quad t \in [0,T].$$

where $\mathbf{w} : \Omega(t) \to \mathbb{R}^d$ is the material velocity of the particles.

Model problem (an example)

Consider a smooth motion and deformation of the material volume $\Omega(t)$, e.g., volume of fluid. For $y \in \Omega_0$, Lagrangian mapping $\Psi(t,y)$ solves

$$\Psi(0,y) = y, \quad \frac{\partial \Psi(t,y)}{\partial t} = \mathbf{w}(t, \Psi(t,y)), \quad t \in [0,T].$$

where $\mathbf{w} : \Omega(t) \to \mathbb{R}^d$ is the material velocity of the particles.

The conservation of a scalar quantity u with a diffusive flux in $\Omega(t)$ solves

$$\begin{aligned} \dot{u} + \mathrm{div}(\mathbf{w})u - \alpha \Delta u &= 0 \quad \text{on} \;\; \Omega(t), \;\; t \in (0,T], \\ \nabla u \cdot \mathbf{n} &= 0 \quad \text{on} \;\; \Gamma(t), \;\; t \in (0,T]. \end{aligned}$$

Model problem (an example)

Consider a smooth motion and deformation of the material volume $\Omega(t)$, e.g., volume of fluid. For $y \in \Omega_0$, Lagrangian mapping $\Psi(t, y)$ solves

$$\Psi(0, y) = y, \quad \frac{\partial \Psi(t, y)}{\partial t} = \mathbf{w}(t, \Psi(t, y)), \quad t \in [0, T].$$

where $\mathbf{w} : \Omega(t) \to \mathbb{R}^d$ is the material velocity of the particles.

The conservation of a scalar quantity u with a diffusive flux in $\Omega(t)$ solves

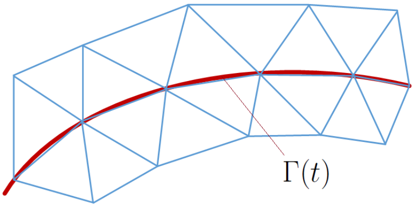
$$\begin{aligned} \dot{u} + \operatorname{div}(\mathbf{w})u - \alpha \Delta u = 0 \quad &\text{on} \quad \Omega(t), \quad t \in (0, T], \\ \nabla u \cdot \mathbf{n} = 0 \quad &\text{on} \quad \Gamma(t), \quad t \in (0, T]. \end{aligned}$$

Warning

In practice Ψ may not be available. Instead, $\Omega(t_n)$ is given in time instances $t_n \in [0, T]$.

To fit or not to fit?

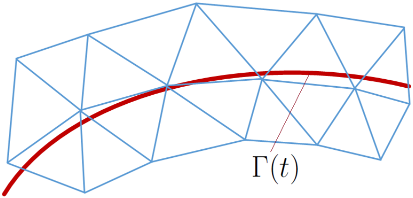
Fitted mesh



Good if:

- Small deformations of $\Omega(t)$
- $\Psi(t) : \Omega_0 \to \Omega(t)$ is available
- Layer adapted mesh is needed

Unfitted mesh



Good if:

- Large deformations
- Geometrical singularities occur
- $\Omega(t)$ is given implicitly
- Cartesian meshes

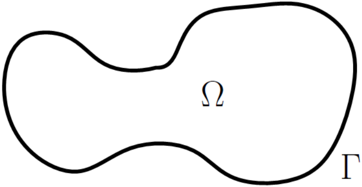
Unfitted FEMs

- Diffuse interface approaches: Immersed interface and immersed boundary methods; Peskin (1977, 2002), ... also for FEMs
- Sharp interface approaches: Partition of Unity FEM, XFEM, cutFEM, TraceFEM; Barrett & Elliott (1987), Melenk & Babuska (1996), Belytschko et al. (1999), Hansbo et al (2002), ...

Unfitted FEMs

Barrett & Elliott unfitted FEM for Neumann problem

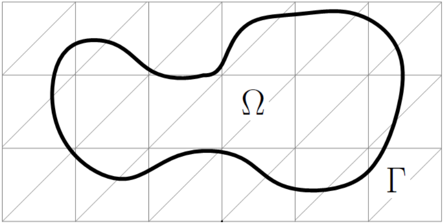
$$
\begin{aligned}
-\Delta u + \alpha\, u &= f \quad \text{in } \Omega,\\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \Gamma,
\end{aligned}
$$



Unfitted FEMs

Barrett & Elliott unfitted FEM for Neumann problem

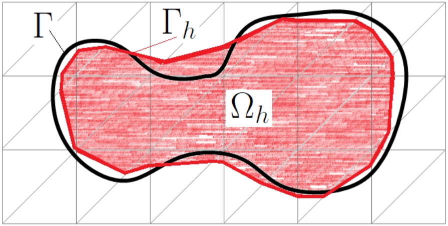
$$-\Delta u + \alpha\, u = f \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma,$$



Unfitted FEMs

Barrett & Elliott unfitted FEM for Neumann problem

$$-\Delta u + \alpha\, u = f \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma,$$



The unfitted FEM:
Find $u_h \in V_h$ satisfying

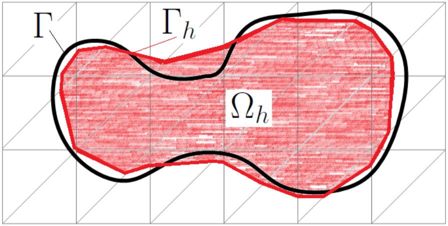
$$\int_{\Omega_h} [\nabla u_h \cdot \nabla v_h + \alpha^e\, u_h v_h]\, d\mathbf{x} = \int_{\Omega_h} f^e v_h\, d\mathbf{x} \quad \forall\, v_h \in V_h,$$

with

$$V_h := \{v \in C(\mathcal{T}_h) \;:\; v|_T \in P_r(T) \quad \forall\, T \in \mathcal{T}_h\}.$$

Unfitted FEMs

Barrett & Elliott unfitted FEM for Neumann problem



$$-\Delta u + \alpha\, u = f \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma,$$

The unfitted FEM:
Find $u_h \in V_h$ satisfying

$$\int_{\Omega_h} [\nabla u_h \cdot \nabla v_h + \alpha^e\, u_h v_h]\, d\mathbf{x} = \int_{\Omega_h} f^e v_h\, d\mathbf{x} \quad \forall\, v_h \in V_h,$$

with

$$V_h := \{v \in C(\mathcal{T}_h) \,:\, v|_T \in P_r(T) \quad \forall\, T \in \mathcal{T}_h\}.$$

From B.-E. and later papers:

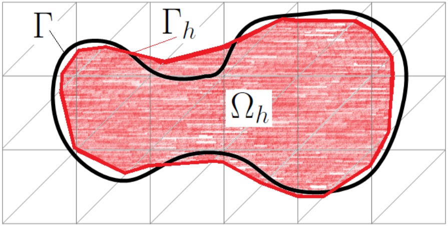
$$\|u^e - u_h\|_{L^2(\Omega_h)} + h\|u^e - u_h\|_{H^1(\Omega_h)} \le C\,(h^{r+1} + h^{q+1}),$$

where $q+1$ is the order of geometry recovery.

Unfitted FEMs

Barrett & Elliott unfitted FEM for
Neumann problem

$$-\Delta u + \alpha\, u = f \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma,$$



The unfitted FEM:
Find $u_h \in V_h$ satisfying

$$\int_{\Omega_h} [\nabla u_h \cdot \nabla v_h + \alpha^e\, u_h v_h]\, d\mathbf{x} = \int_{\Omega_h} f^e v_h\, d\mathbf{x} \quad \forall\, v_h \in V_h,$$

with

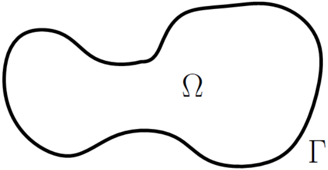
$$V_h := \{v \in C(\mathcal{T}_h) \;:\; v|_T \in P_r(T) \quad \forall\, T \in \mathcal{T}_h\}.$$

- Dirichlet's b.c.?
- Algebraic stability?

Unfitted FEMs

Unfitted FEM for Dirichlet problem

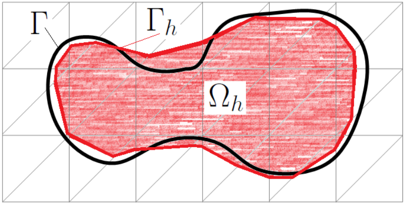
$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma, \end{aligned}$$



Unfitted FEMs

Unfitted FEM for Dirichlet problem

$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = g \quad \text{on } \Gamma,$$



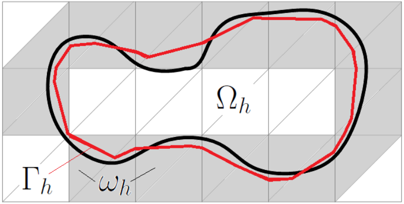
The unfitted FEM + Nitsche

$$\int_{\Omega_h} \nabla u_h \cdot \nabla v_h \, d\mathbf{x} - \int_{\Gamma_h} \frac{\partial u_h}{\partial n} v_h \, d\mathbf{s} - \int_{\Gamma_h} \frac{\partial v_h}{\partial n} (u_h - g^e) \, d\mathbf{s}$$
$$+\gamma_D \int_{\Gamma_h} h^{-1} v_h (u_h - g^e) \, d\mathbf{s} \qquad = \int_{\Omega_h} f^e v_h \, d\mathbf{x} \quad \forall \, v_h \in V_h.$$

Unfitted FEMs

Unfitted FEM for Dirichlet problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma, \end{aligned}$$



The unfitted FEM + Nitsche +stabilization

$$\int_{\Omega_h} \nabla u_h \cdot \nabla v_h \, d\mathbf{x} - \int_{\Gamma_h} \frac{\partial u_h}{\partial n} v_h \, d\mathbf{s} - \int_{\Gamma_h} \frac{\partial v_h}{\partial n} (u_h - g^e) \, d\mathbf{s}$$

$$+\gamma_D \int_{\Gamma_h} h^{-1} v_h (u_h - g^e) \, d\mathbf{s} + j(u_h, v_h) = \int_{\Omega_h} f^e v_h \, d\mathbf{x} \quad \forall\, v_h \in V_h.$$

with

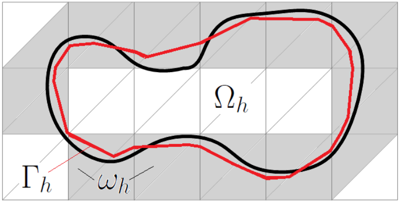
$$j(u_h, v_h) = \sum_{E \in \omega_h} \int_{\Gamma_h} h [\![\partial_{\mathbf{n}} v_h]\!] [\![\partial_{\mathbf{n}} u_h]\!] \, d\mathbf{s} \quad \text{for } P_1 \text{ FEM}.$$

Burman & Hansbo (2014)

Unfitted FEMs

Unfitted FEM for Dirichlet problem

$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = g \quad \text{on } \Gamma,$$



The unfitted FEM + Nitsche +stabilization

$$\int_{\Omega_h} \nabla u_h \cdot \nabla v_h \, d\mathbf{x} - \int_{\Gamma_h} \frac{\partial u_h}{\partial n} v_h \, d\mathbf{s} - \int_{\Gamma_h} \frac{\partial v_h}{\partial n}(u_h - g^e)\, d\mathbf{s}$$
$$+\gamma_D \int_{\Gamma_h} h^{-1} v_h (u_h - g^e)\, d\mathbf{s} + j(u_h, v_h) = \int_{\Omega_h} f^e v_h \, d\mathbf{x} \quad \forall\, v_h \in V_h.$$

with

$$j(u_h, v_h) = \sum_{E \in \omega_h} \sum_{j=1}^{r} \int_E \frac{h^{2j-1}}{k!^2} [\![D^j v_h]\!] [\![D^j u_h]\!] \, ds \quad \text{for } P_r \text{ FEM.}$$

Burman & Hansbo (2014)

Unfitted FEM: time-dependent $\Omega(t)$

$$\begin{aligned} \dot{u} + \operatorname{div}(\mathbf{w})u - \Delta u = 0 \quad &\text{on} \quad \Omega(t), \quad t \in (0,T], \\ \nabla u \cdot \mathbf{n} = 0 \quad &\text{on} \quad \Gamma(t), \quad t \in (0,T]. \end{aligned}$$

Unfitted FEM: time-dependent $\Omega(t)$

$$\begin{aligned} \dot{u} + \mathrm{div}(\mathbf{w})u - \Delta u = 0 \quad &\text{on} \;\; \Omega(t), \quad t \in (0,T], \\ \nabla u \cdot \mathbf{n} = 0 \quad &\text{on} \;\; \Gamma(t), \quad t \in (0,T]. \end{aligned}$$

Space–time weak formulation:

$$\int_0^T \int_{\Omega(t)} \{\frac{\partial u}{\partial t} + \mathrm{div}(\mathbf{w}u)\} v \, dx \, dt + \int_0^T \int_{\Omega(t)} \nabla u \cdot \nabla v \, dx \, dt = 0.$$

Unfitted FEM: time-dependent $\Omega(t)$

$$\dot{u} + \mathrm{div}(\mathbf{w})u - \Delta u = 0 \quad \text{on} \quad \Omega(t), \quad t \in (0, T],$$
$$\nabla u \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma(t), \quad t \in (0, T].$$

Space–time weak formulation:

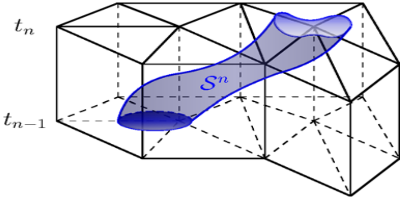
$$\int_0^T \int_{\Omega(t)} \{\frac{\partial u}{\partial t} + \mathrm{div}(\mathbf{w}u)\} v \, dx \, dt + \int_0^T \int_{\Omega(t)} \nabla u \cdot \nabla v \, dx \, dt = 0.$$

Space–time unfitted (XFEM/cutFEM) FE + stabilization

$$\int_{S_h^n} \{\frac{\partial u_h}{\partial t} + \mathrm{div}(\mathbf{w}u_h)\} v_h \, d\mathbf{x} + \int_{S_h^n} \nabla u_h \cdot \nabla v_h \, d\mathbf{x} + \int_{\Omega_h^{n-1}} [u_h] v_h^+ \, dx + j_n(u_h, v_h) = 0.$$

with

$$S_h^n = \bigcup_{t \in (t_{n-1}, t_n)} \Omega_h(t) \times \{t\}.$$



Hansbo et al (2016), Lehrenfeld, Reusken (2013, 2015)

Method of lines for unfitted FEM on moving domains

Implicit Euler without care

$$\frac{u^n - \; u^{n-1}}{\Delta t} + \mathrm{div}(u^n \mathbf{w}^n) - \alpha \Delta u^n = 0 \qquad \text{on} \;\; \Omega^n.$$

- But u^{n-1} may not be defined on Ω^n

Method of lines for unfitted FEM on moving domains

Implicit Euler with more care

$$\frac{u^n - \mathcal{E}u^{n-1}}{\Delta t} + \mathrm{div}(u^n \mathbf{w}^n) - \alpha \Delta u^n = 0 \qquad \text{on} \quad \Omega^n.$$

- But u^{n-1} may not be defined on Ω^n
- extend u^{n-1} with $\mathcal{E} : H^1(\Omega^{n-1}) \to H^1(U_\delta(\Omega^{n-1}))$, $U_\delta(\Omega^{n-1}) \supset \Omega^n$

Method of lines for unfitted FEM on moving domains

Implicit Euler with more care

$$\frac{u^n - \mathcal{E}u^{n-1}}{\Delta t} + \mathrm{div}(u^n \mathbf{w}^n) - \alpha \Delta u^n = 0 \qquad \text{on} \quad \Omega^n.$$

- But u^{n-1} may not be defined on Ω^n
- extend u^{n-1} with $\mathcal{E}: H^1(\Omega^{n-1}) \to H^1(U_\delta(\Omega^{n-1}))$, $U_\delta(\Omega^{n-1}) \supset \Omega^n$
- $\to$ stability of semi-discretization

Key technical result (uses *uniform* continuity of $\mathcal{E}$)

$$\|\mathcal{E}u\|^2_{U_\delta(\Omega^n)} \le (1+c\,(1+\varepsilon^{-1})\,\delta)\,\|u\|^2_{\Omega^n} + c\,\delta\,\varepsilon\|\nabla u\|^2_{\Omega^n} \quad \forall u \in H^1(\Omega^n),\ \varepsilon > 0.$$

Implies ‘energy’ stability bound for the semi-discrete method.

Method of lines for unfitted FEM on moving domains

Implicit Euler with more care

$$\frac{u^n - \mathcal{E}u^{n-1}}{\Delta t} + \mathrm{div}(u^n \mathbf{w}^n) - \alpha \Delta u^n = 0 \qquad \text{on} \;\; \Omega^n.$$

- But u^{n-1} may not be defined on Ω^n
- extend u^{n-1} with $\mathcal{E} : H^1(\Omega^{n-1}) \to H^1(U_\delta(\Omega^{n-1}))$, $U_\delta(\Omega^{n-1}) \supset \Omega^n$
- $\to$ stability of semi-discretization
- **How to realize a FE extension ?**

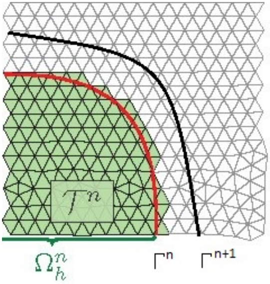
Key technical result (uses *uniform* continuity of $\mathcal{E}$)

$$\|\mathcal{E}u\|^2_{U_\delta(\Omega^n)} \leq (1+c\,(1+\varepsilon^{-1})\,\delta)\,\|u\|^2_{\Omega^n} + c\,\delta\,\varepsilon\|\nabla u\|^2_{\Omega^n} \quad \forall u \in H^1(\Omega^n),\ \varepsilon > 0.$$

Implies ‘energy’ stability bound for the semi-discrete method.

Implicit FE extensions

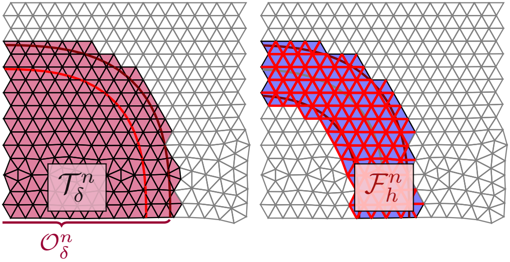
Extension from $\mathcal{T}^n$



- stepping from t^{n-1} to t^n gives equations for unknowns in "active mesh" $\mathcal{T}^n$

Implicit FE extensions

Implicit extension from $\mathcal{T}^n$ to $\mathcal{T}_\delta^n$

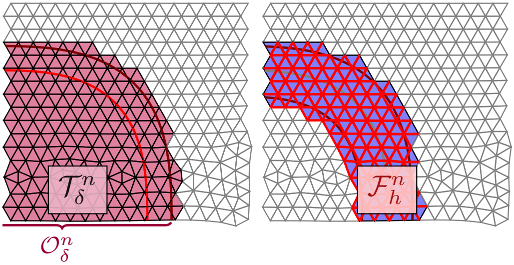


- stepping from t^{n-1} to t^n gives equations for unknowns in $\mathcal{T}^n$
- add ghost penalty stabilization in a δ-layer around Γ_h^n:

$$j_h^{\text{djump}}(u_h, v_h) = \sum_{E \in \mathcal{F}_h^n} \sum_{j=1}^{r} \int_E \frac{h^{2j-1}}{j!^2} [\![D^j v_h]\!][\![D^j u_h]\!]$$

Implicit FE extensions

Implicit extension from $\mathcal{T}^n$ to $\mathcal{T}^n_\delta$

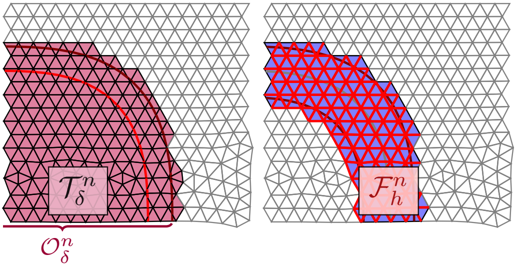


- stepping from t^{n-1} to t^n gives equations for unknowns in $\mathcal{T}^n$
- add ghost penalty stabilization in a δ-layer around Γ^n_h:

$$j_h^{\mathrm{dir}}(u_h, v_h) = \sum_{E\in\mathcal{F}^n_h} \int_{\omega_E} \frac{1}{h^2} [\![u]\!]_{\omega_E} [\![v]\!]_{\omega_E}\, dx$$

Implicit FE extensions

Implicit extension from $\mathcal{T}^n$ to $\mathcal{T}_\delta^n$

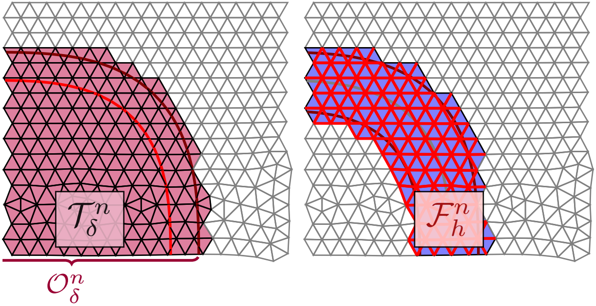


- stepping from t^{n-1} to t^n gives equations for unknowns in $\mathcal{T}^n$
- add ghost penalty stabilization in a δ-layer around Γ_h^n:

$$j_h^{\text{lps}}(u_h, v_h) = \sum_{E \in \mathcal{F}_h^n} \frac{1}{h^2} \int_{\omega_E} (u - \Pi_{\omega_E} u)(v - \Pi_{\omega_F} v) dx$$

Implicit FE extensions

Implicit extension from $\mathcal{T}^n$ to $\mathcal{T}_\delta^n$



- stepping from t^{n-1} to t^n gives equations for unknowns in $\mathcal{T}^n$
- add ghost penalty stabilization in a δ-layer around Γ_h^n:
- choose δ sufficiently large so that $\mathcal{T}_\delta^n \supset \Omega_h^{n+1}$: $\delta \approx |\mathbf{w} \cdot \mathbf{n}|\Delta t$.

Variational formulation of an implicit Euler step

$$\int_{\Omega_h^n} \frac{u_h^n - u_h^{n-1}}{\Delta t} v_h + \underbrace{a_h^n(u_h^n, v_h)}_{\text{convection diffusion}} + j_h^n(u_h^n, v_h) = 0 \quad \text{for all } v_h \in V_h^n;$$

- u_h^n is well-defined in $\mathcal{T}_\delta^n$ such that $\Omega_h^n, \Omega_h^{n+1} \subset \mathcal{T}_\delta^n$

Variational formulation of an implicit Euler step

$$\int_{\Omega_h^n} \frac{u_h^n - u_h^{n-1}}{\Delta t} v_h + \underbrace{a_h^n(u_h^n, v_h)}_{\text{convection diffusion}} + j_h^n(u_h^n, v_h) = 0 \quad \text{for all } v_h \in V_h^n;$$

- u_h^n is well-defined in $\mathcal{T}_\delta^n$ such that $\Omega_h^n, \Omega_h^{n+1} \subset \mathcal{T}_\delta^n$
- Control on the implicit extension (on V_h):

$$\|u_h\|^2_{\mathcal{O}_\delta(\Omega_h^n)} \leq (1+c\Delta t)\|u_h\|^2_{\Omega_h^n} + c\Delta t\|\nabla u_h\|^2_{\Omega_h^n} + c\Delta t j_h^n(u_h, u_h)$$

Variational formulation of an implicit Euler step

$$\int_{\Omega_h^n} \frac{u_h^n - u_h^{n-1}}{\Delta t} v_h + \underbrace{a_h^n(u_h^n, v_h)}_{\text{convection diffusion}} + j_h^n(u_h^n, v_h) = 0 \quad \text{for all } v_h \in V_h^n;$$

- u_h^n is well-defined in $\mathcal{T}_\delta^n$ such that $\Omega_h^n, \Omega_h^{n+1} \subset \mathcal{T}_\delta^n$
- Control on the implicit extension (on V_h):

$$\|u_h\|^2_{\mathcal{O}_\delta(\Omega_h^n)} \leq (1+c\Delta t)\|u_h\|^2_{\Omega_h^n} + c\Delta t\|\nabla u_h\|^2_{\Omega_h^n} + c\Delta t j_h^n(u_h, u_h)$$

- Consistency: $j_h^n(\mathcal{E}u, v) \lesssim h^r \|u\|_{H^{r+1}(\Omega^n)} j_h^n(v, v)^{\frac{1}{2}}$.

Variational formulation of an implicit Euler step

$$\int_{\Omega_h^n} \frac{u_h^n - u_h^{n-1}}{\Delta t} v_h + \underbrace{a_h^n(u_h^n, v_h)}_{\text{convection diffusion}} + j_h^n(u_h^n, v_h) = 0 \quad \text{for all } v_h \in V_h^n;$$

- u_h^n is well-defined in $\mathcal{T}_\delta^n$ such that $\Omega_h^n, \Omega_h^{n+1} \subset \mathcal{T}_\delta^n$
- Control on the implicit extension (on V_h):

$$\|u_h\|^2_{\mathcal{O}_\delta(\Omega_h^n)} \le (1+c\Delta t)\|u_h\|^2_{\Omega_h^n} + c\Delta t\|\nabla u_h\|^2_{\Omega_h^n} + c\Delta t j_h^n(u_h, u_h)$$

- Consistency: $j_h^n(\mathcal{E}u, v) \lesssim h^r \|u\|_{H^{r+1}(\Omega^n)} j_h^n(v,v)^{\frac{1}{2}}$.

Convergence result (here implicit Euler)

$$\|u(t^n) - u_h^n\|^2_{\Omega_h^n} + \sum_{k=1}^{n} \|\nabla(u(t^k) - u_h^k)\|^2_{\Omega_k^n}$$
$$\lesssim \exp(ct_n)R(u)(\underbrace{|\Delta t|^2}_{\text{time}} + \underbrace{h^{2q}}_{\text{geometry approx.}} + \underbrace{h^{2r}}_{\text{space}} \cdot \underbrace{(1 + \Delta t/h))}_{\text{anisotropy in space-time}}$$

New approach

- Simple: only requires spatial integrals and FE spaces

New approach

- Simple: only requires spatial integrals and FE spaces
  - More straightforward to implement than space–time FEM

New approach

- Simple: only requires spatial integrals and FE spaces
  - More straightforward to implement than space–time FEM
  - Does not require PDE extension as in classical fictitious domain method

New approach

- Simple: only requires spatial integrals and FE spaces
- Higher order in time is possible (e.g. BDF2 is straightforward)

New approach

- Simple: only requires spatial integrals and FE spaces
- Higher order in time is possible (e.g. BDF2 is straightforward)

	1	2	3	4	5	6	7	$\mathrm{eoc_t}$
0	$1.2 \cdot 10^{-2}$	$7.6 \cdot 10^{-3}$	$6.0 \cdot 10^{-3}$	$5.3 \cdot 10^{-3}$	$5.0 \cdot 10^{-3}$	$4.9 \cdot 10^{-3}$	$4.8 \cdot 10^{-3}$	—
1	$6.3 \cdot 10^{-3}$	$3.6 \cdot 10^{-3}$	$2.7 \cdot 10^{-3}$	$2.4 \cdot 10^{-3}$	$2.2 \cdot 10^{-3}$	$2.1 \cdot 10^{-3}$	$2.0 \cdot 10^{-3}$	1.24
2	$4.6 \cdot 10^{-3}$	$1.7 \cdot 10^{-3}$	$9.1 \cdot 10^{-4}$	$7.0 \cdot 10^{-4}$	$6.4 \cdot 10^{-4}$	$6.1 \cdot 10^{-4}$	$6.0 \cdot 10^{-4}$	1.76
3	$3.8 \cdot 10^{-3}$	$1.2 \cdot 10^{-3}$	$3.9 \cdot 10^{-4}$	$2.1 \cdot 10^{-4}$	$1.7 \cdot 10^{-4}$	$1.7 \cdot 10^{-4}$	$1.7 \cdot 10^{-4}$	1.86
4	$3.6 \cdot 10^{-3}$	$9.4 \cdot 10^{-4}$	$2.7 \cdot 10^{-4}$	$9.1 \cdot 10^{-5}$	$5.2 \cdot 10^{-5}$	$4.6 \cdot 10^{-5}$	$4.6 \cdot 10^{-5}$	1.85
5	$3.5 \cdot 10^{-3}$	$9.0 \cdot 10^{-4}$	$2.3 \cdot 10^{-4}$	$6.3 \cdot 10^{-5}$	$2.2 \cdot 10^{-5}$	$1.4 \cdot 10^{-5}$	$1.2 \cdot 10^{-5}$	1.88
6	$3.5 \cdot 10^{-3}$	$8.8 \cdot 10^{-4}$	$2.2 \cdot 10^{-4}$	$5.6 \cdot 10^{-5}$	$1.5 \cdot 10^{-5}$	$5.5 \cdot 10^{-6}$	$3.5 \cdot 10^{-6}$	1.82
7	$3.5 \cdot 10^{-3}$	$8.8 \cdot 10^{-4}$	$2.2 \cdot 10^{-4}$	$5.4 \cdot 10^{-5}$	$1.4 \cdot 10^{-5}$	$3.8 \cdot 10^{-6}$	$1.4 \cdot 10^{-6}$	1.36
$\mathrm{eoc_x}$	1.91	1.98	2.01	2.00	1.98	1.87	1.46	
$\mathrm{eoc_{xt}}$	1.77	1.89	2.13	2.10	2.04	2.02	2.01	

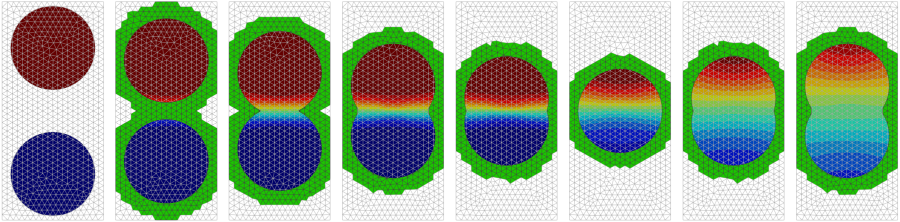
$L^2(L^2)$ error for the BDF2 method for translated circle

New approach

- Simple: only requires spatial integrals and FE spaces
- Higher order in time is possible (e.g. BDF2 is straightforward)
- Sharp interface method; Easy to combine with implicit definitions of $\Omega(t)$, i.e. level-set method;

New approach

- Simple: only requires spatial integrals and FE spaces
- Higher order in time is possible (e.g. BDF2 is straightforward)
- Sharp interface method; Easy to combine with implicit definitions of $\Omega(t)$, i.e. level-set method;
- Robust even for topology changes:



$t = 0.0T$ $t = 0.1T$ $t = 0.2T$ $t = 0.3T$ $t = 0.4T$ $t = 0.5T$ $t = 0.6T$ $t = 0.7T$

New approach

- Simple: only requires spatial integrals and FE spaces
- Higher order in time is possible (e.g. BDF2 is straightforward)
- Sharp interface method; Easy to combine with implicit definitions of $\Omega(t)$, i.e. level-set method;
- Robust even for topology changes:
- Complete error analysis is available (in energy norm), including geometrical error bounds.

"Thank you" slide

Further details: C. Lehrenfeld, M. Olshanskii, An Eulerian Finite Element Method for PDEs in time-dependent domains, arXiv:1803.01779

"Thank you" slide

Further details: C. Lehrenfeld, M. Olshanskii, An Eulerian Finite Element Method for PDEs in time-dependent domains, arXiv:1803.01779

Predecessor paper: C. Lehrenfeld, M. Olshanskii, X Xu, A stabilized trace finite element method for partial differential equations on evolving surfaces, SINUM, (2018).

"Thank you" slide

Further details: C. Lehrenfeld, M. Olshanskii, An Eulerian Finite Element Method for PDEs in time-dependent domains, arXiv:1803.01779

Predecessor paper: C. Lehrenfeld, M. Olshanskii, X Xu, A stabilized trace finite element method for partial differential equations on evolving surfaces, SINUM, (2018).

Important papers:

- J. W. Barrett, C.M. Elliott, Fitted and unfitted finite-element methods for elliptic equations with smooth interfaces. IMA journal of numerical analysis, (1987).
- E. Burman, Ghost penalty, C. R. Math. Acad. Sci. Paris, (2010)
- E. Burman and P. Hansbo, Fictitious domain finite element methods using cut elements: I. & II. (2010, 2012)

"Thank you" slide