Maximum principle in multiphase flow problems

Novikov Konstantin

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Outline

Two-phase flow model

$$\frac{\partial}{\partial t}\left(\frac{\phi s_\alpha}{b_\alpha}\right) + div\left(\frac{1}{b_\alpha}u_\alpha\right) = q_\alpha,$$

$$u_\alpha = -\frac{k_{r\alpha}}{\mu_\alpha}\mathbb{K}(\nabla p_\alpha - \rho_\alpha g \nabla z), \alpha = w, o$$

$$p_o - p_w = p_c(s_w), s_o + s_w = 1$$

- $\alpha = w, o$ – phase (water, oil)
- s_α – saturation
- p_α – pressure
- u_α – Darcy velocity
- b_α – formation factor
- q_α – sources

+ initial and boundary conditions

- p_c – capillary pressure
- μ_α – viscosity
- $k_{r\alpha}$ – relative permeability
- $\mathbb{K}$ – absolute permeability
- ρ_α – density

Three-phase flow model

$$\frac{\partial}{\partial t}\left(\frac{\phi s_\alpha}{b_\alpha}\right) + div\left(\frac{1}{b_\alpha}u_\alpha\right) = q_\alpha, \alpha = w, o,$$
$$\frac{\partial}{\partial t}\left(\frac{\phi s_g}{b_g} - \frac{r_s s_o}{b_o}\right) + div(\frac{1}{b_g}u_g + \frac{r_s}{b_o}u_o) = q_g,$$
$$u_\alpha = -\frac{k_{r\alpha}}{\mu_\alpha}\mathbb{K}(\nabla p_\alpha - \rho_\alpha g \nabla z), \alpha = w, o, g$$
$$s_w + s_o + s_g = 1, p_\alpha - p_o = p_{c\alpha}, \alpha = w, o, g,$$

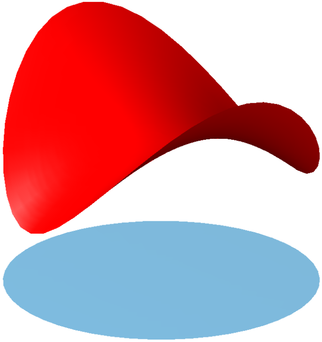
- $\alpha = w, o, g$ – water, oil, gas
- $p_{c\alpha}$ – capilary pressure
- r_s – gas solubility

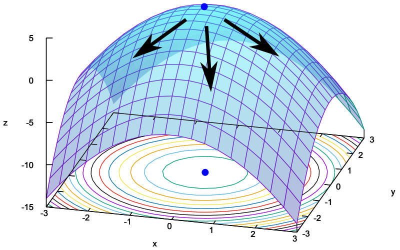
- s_α – saturation
- p_α – pressure
- u_α – Darcy velocity
- b_α – formation factor
- q_α – sources

+ initial and boundary conditions

- μ_α – viscosity
- $k_{r\alpha}$ – rel. permeability
- $\mathbb{K}$ – abs. permeability
- ρ_α – density

Maximum principle



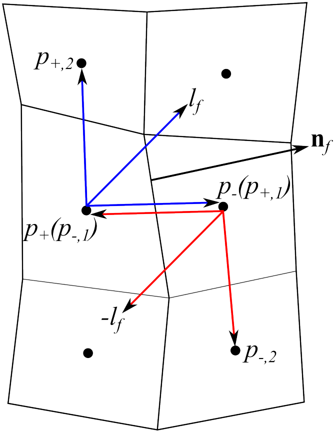


Differential maximum principles: summary

	two-phase			three-phase	
	p_α	$p_o + \widetilde{p}$	s_α	p_α	$p_o + \widetilde{p}$
$p_{cwo} \equiv 0, p_{cgo} \equiv 0$	required	not required	not required	required	not required
$\exists \widetilde{p} : \nabla \widetilde{p} = f_w \nabla p_{cwo} + f_g \nabla p_{cgo}$	required	required	not required	required	required
$\mu_\alpha \equiv const$	not required	required	required	not required	required
$b_\alpha \equiv 1$	not required	not required	required	not required	not required
$\phi \equiv const$	not required	not required	required	not required	not required

- ■ (green) required by a theorem
- ■ (red) not required by a theorem

Finite volume scheme



$$\mathbb{K}n_f = l_f = \alpha_1 \overrightarrow{P_+P_{+1}} + \alpha_2 \overrightarrow{P_+P_{+2}}$$
$$-\mathbb{K}n_f = -l_f = \beta_1 \overrightarrow{P_+P_{-1}} + \beta_2 \overrightarrow{P_+P_{-2}}$$

$$q_1 = -\mathbb{K}\nabla p \cdot n_f = \nabla p \cdot l_f =$$
$$-\nabla p \left(\alpha_1 \overrightarrow{P_+P_{+1}} + \alpha_2 \overrightarrow{P_+P_{+2}}\right) =$$
$$\alpha_1 (p_{+,1} - p_+) + \alpha_2 (p_{+,2} - p_+)$$

$$q = \mu_1 q_1 + \mu_2(-q_2)$$
$$\mu_1 q_1 + \mu_2 q_2 = 0$$
$$\mu_1 + \mu_2 = 1$$

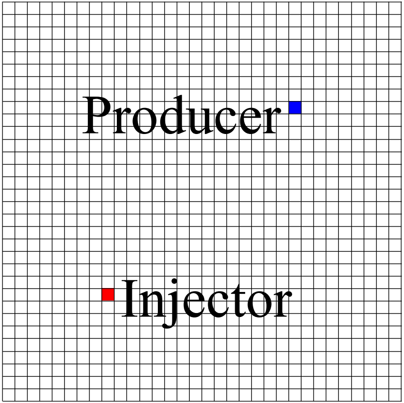
[*Lipnikov K., Svyatskiy D., Vassilevsky Yu.* Minimal stencil finite volume scheme with the discrete maximum principle // Russian Journal of Numerical Analysis and Mathematical Modelling. 27(4). 2012.]

Differential and discrete maximum principles for pressure in two-phase flow model

	Differential	Discrete
$p_c \equiv 0$	🟩	🟩
$\mu_\alpha \equiv const$	🟥	🟥
$b_\alpha \equiv 1$	🟥	🟩
$\phi \equiv const$	🟥	🟩

- 🟩 required by a theorem
- 🟥 not required by a theorem

Numerical experiment #1



1. zero capillary pressure
   $p_c \equiv 0$
2. constant viscosities
   $\mu_\alpha = const$
3. incompressibility $b_\alpha = 1$
4. constant porosity $\phi \equiv const$
5. Absolute permeability $\mathbb{K} = R_z(-\theta_z) diag(k_1, k_2, k_3) R_z(\theta_z)$, where
   - $k_1 = k_3 = 100$, $k_2 = 0.1$,
   - $\theta_z = 112.5°$,
   - $R_z(\alpha)$ is the matrix of rotation in xy-plane.

Numerical pressures

	multi-point	nonlinear two-point	linear two-point
max	4099.81	4112.64	4097.03
min	3901.10	3888.20	3928.91

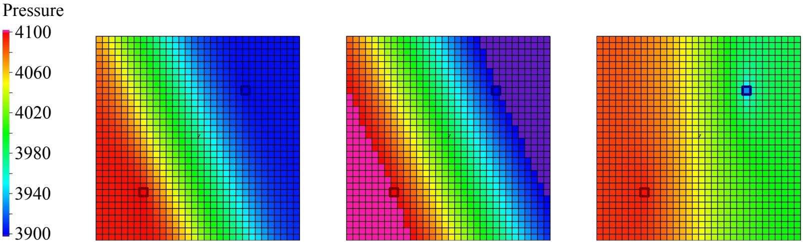


Рис.: Pressure after 2000 model days for different flux discretization schemes.

Experimental discrete maximum principle for nonconstant parameters

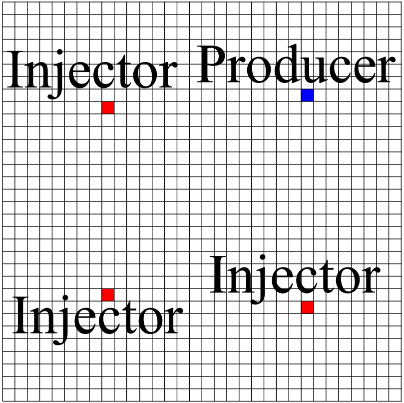
	Differential	Discrete
$p_c \equiv 0$	■	■
$\mu_\alpha \equiv const$	■	■
$b_\alpha \equiv 1$	■	■
$\phi \equiv const$	■	■

- required by a theorem
- not required by a theorem

Summary

- 3 differential maximum principles for two-phase flow model and 2 for three-phase flow model have been proven.
- The discrete maximum principle for numerical pressure obtained using nonlinear multipoint scheme has been proven.
- The discrete maximum principle require additional assumption on model coefficients.
- Numerical experiments support possible existence of the discrete maximum principle for fewer assumptions.

Numerical experiment #2



1. zero capillary pressure $p_c \equiv 0$
2. constant viscosities $\mu_\alpha = const$
3. incompressibility $b_\alpha = 1$
4. constant porosity $\phi \equiv const$
5. Absolute permeability $\mathbb{K} = R_z(-\theta_z) diag(k_1, k_2, k_3) R_z(\theta_z)$, where
   - $k_1 = k_3 = 100, k_2 = 0.1$,
   - $\theta_z = 112.5^\circ$,
   - $R_z(\alpha)$ is the matrix of rotation in xy-plane.

Numerical pressures

	multi-point	nonlinear two-point	linear two-point
max	4099.91	4122.31	4099.07
min	3906.32	3852.74	3952.38

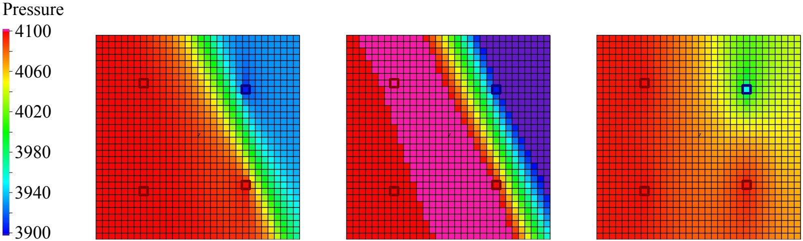


Рис.: Pressure after 100 model days for different flux discretization schemes.

Numerical saturations

	multi-point	nonlinear two-point	linear two-point
max	0.517	0.510	0.574
min	0.150	-0.995	0.150

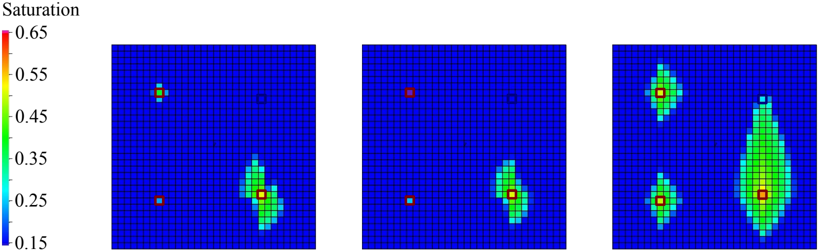


Рис.: Water saturation after 100 model days for different flux discretization schemes. Initial saturation is $s(0) = 0.15$.

Two-phase flow model equations (no gravity)

$$\begin{cases} \frac{\partial}{\partial t}\left(\frac{\phi\rho_\alpha s_\alpha}{b_\alpha}\right) - div\left(\frac{\rho_\alpha}{b_\alpha}\frac{k_{r\alpha}}{\mu_\alpha}\mathbb{K}\nabla p_\alpha\right) = q_\alpha \text{ in } \Omega \times (0, T) \\ p_o - p_w = p_c(s_w) \\ s_o + s_w = 1 \end{cases}$$

Differential maximum principle. Assumptions(1).

- zero capillary pressure $p_c \equiv 0$
- strictly elliptic absolute permeability $\mathbb{K}$
- smooth enough $b_\alpha, \lambda_\alpha, \alpha = w, o$

- no incompressibility assumption: $b_\alpha \not\equiv const$
- no constant porosity assumption: $\phi \not\equiv const$
- no constant viscosity assumption: $\mu_\alpha \not\equiv const$

Differential maximum principle(1). Pressure.

- $b_o q_o + b_w q_w \leq 0$ in $\Omega \times [0, T] \Rightarrow$

$$\sup_{\Omega \times [0,T]} p_\alpha \leq \sup_{\partial\Omega \times [0,T)} p_\alpha$$

- $b_o q_o + b_w q_w \geq 0$ in $\Omega \times [0, T] \Rightarrow$

$$\inf_{\Omega \times [0,T]} p_\alpha \geq \inf_{\partial\Omega \times [0,T)} p_\alpha, \alpha = w, o$$

Differential maximum principle. Assumptions(2)

- fractional flows $f_\alpha = \frac{\lambda_\alpha}{\lambda_w + \lambda_o}$, $\alpha = w, o$ depend solely on s_w
  - (implies constant viscosities, since $\lambda_\alpha = \frac{k_{r\alpha}}{\mu_\alpha}$ and $\mu_\alpha = \mu_\alpha(p_\alpha)$)
- there exists function $\widetilde{p}$ such that $\nabla \widetilde{p} = f_w \nabla p_c$[1],
- strictly elliptic absolute permeability $\mathbb{K}$
- smooth enough $b_\alpha, \lambda_\alpha, \alpha = w, o$

- no incompressibility assumption: $b_\alpha \not\equiv const$
- no constant porosity assumption: $\phi \not\equiv const$

[1] Chen Z. Formulations and Numerical Methods of the Black Oil Model in Porous Media. SIAM J. Numer. Anal., 2000; 38(2):489–514.

Differential maximum principle(2). Pressure.

- $b_o q_o + b_w q_w \leq 0$ in $\Omega \times [0, T] \Rightarrow$

$$\sup_{\Omega\times[0,T]} p \leq \sup_{\partial\Omega\times[0,T)} p$$

- $b_o q_o + b_w q_w \geq 0$ in $\Omega \times [0, T] \Rightarrow$

$$\inf_{\Omega\times[0,T]} p \geq \inf_{\partial\Omega\times[0,T)} p$$

where $p = p_o - \widetilde{p}$.

Differential maximum principle. Assumptions(3)

- constant viscosities $\mu_\alpha = const$, $\alpha = w, o$
- incompressibility: $b_\alpha = 1, \alpha = w, o$
- constant porosity: $\phi = const$,
- relative permeabilities $k_{r\alpha}$ are monotonic functions of s_w
- p_c is monotonically decreasing function of s_w
- strictly elliptic absolute permeability $\mathbb{K}$
- smooth enough $k_{r\alpha}, \alpha = w, o$

- no constant capillary pressure assumption: $p_c \not\equiv 0$,

Differential maximum principle(3). Saturations.

- $q_w \le 0, q_o \ge 0$ in $\Omega \times [0, T] \Rightarrow$

$$\sup_{\Omega\times[0,T]} s_w \le \sup_{\partial\Omega\times[0,T)} s_w, \qquad \inf_{\Omega\times[0,t]} s_o \ge \inf_{\partial\Omega\times[0,T)} s_o$$

- $q_w \ge 0, q_o \le 0$ in $\Omega \times [0, T] \Rightarrow$

$$\sup_{\Omega\times[0,T]} s_o \le \sup_{\partial\Omega\times[0,T)} s_o, \qquad \inf_{\Omega\times[0,t]} s_w \ge \inf_{\partial\Omega\times[0,T)} s_w$$

Three-phase flow (no gravity)

$$\begin{cases} \frac{\partial}{\partial t}\left(\frac{\phi\rho_w s_w}{b_w}\right) - div\left(\frac{\rho_w}{b_w}\frac{k_{rw}}{\mu_w}\mathbb{K}\nabla p_w\right) = q_w \\ \frac{\partial}{\partial t}\left(\frac{\phi\rho_o s_o}{b_o}\right) - div\left(\frac{\rho_o}{b_o}\frac{k_{ro}}{\mu_o}\mathbb{K}\nabla p_o\right) = q_o \\ \frac{\partial}{\partial t}\left(\frac{\phi\rho_g s_g}{b_g} + \frac{r_{so}\rho_o s_o}{b_o}\right) - div(\frac{\rho_g}{b_g}\frac{k_{rg}}{\mu_g}\mathbb{K}\nabla p_g + \\ \qquad + \frac{r_{so}\rho_o}{b_o}\frac{k_{ro}}{\mu_o}\mathbb{K}\nabla p_o) = q_g \\ s_o + s_w + s_g = 1 \\ p_\alpha - p_o = p_{c\alpha o}, \alpha = g, w \end{cases}$$

Differential maximum principle. Assumptions(1)

- $p_{c\alpha o} \equiv 0, \alpha = w, g$
- strictly elliptic absolute permeability $\mathbb{K}$
- smooth enough $b_\alpha, \lambda_\alpha, \alpha = w, o$

- no incompressibility assumption: $b_\alpha \not\equiv const$
- no constant porosity assumption: $\phi \not\equiv const$
- no constant viscosity assumption: $\mu_\alpha \not\equiv const$

Differential maximum principle(1). Pressure

- $b_o q_o + b_w q_w + b_g q_g - b_g r_{so} q_o \leq 0$ in $\Omega \times [0, T] \Rightarrow$

$$\sup_{\Omega\times[0,T]} p_\alpha \leq \sup_{\partial\Omega\times[0,T)} p_\alpha$$

- $b_o q_o + b_w q_w + b_g q_g - b_g r_{so} q_o \geq 0$ in $\Omega \times [0, T] \Rightarrow$

$$\inf_{\Omega\times[0,T]} p_\alpha \geq \inf_{\partial\Omega\times[0,T)} p_\alpha, \alpha = w, o, g$$

Differential maximum principle. Assumptions(2)

- fractional flows $f_\alpha = \frac{\lambda_\alpha}{\lambda_w+\lambda_o+\lambda_g}$, $\alpha = w, g, o$ depend solely on s_w and s_o
  - (implies constant viscosities, since $\lambda_\alpha = \frac{k_{r\alpha}}{\mu_\alpha}$ and $\mu_\alpha = \mu_\alpha(p_\alpha)$)
- there exits such function $\widetilde{p}$ that $\nabla\widetilde{p} = f_w \nabla p_{cwo} + f_g \nabla p_{cgo}$[1]
- strictly elliptic absolute permeability $\mathbb{K}$
- smooth enough $b_\alpha, \lambda_\alpha, \alpha = w, o$

- no incompressibility assumption: $b_\alpha \not\equiv const$
- no constant porosity assumption: $\phi \not\equiv const$

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Differential maximum principle(2). Pressure.

- $b_o q_o + b_w q_w + b_g q_g - b_g r_{so} q_o \leq 0$ in $\Omega \times [0, T] \Rightarrow$

$$\sup_{\Omega \times [0,T]} p \leq \sup_{\partial\Omega \times [0,T)} p$$

- $b_o q_o + b_w q_w + b_g q_g - b_g r_{so} q_o \geq 0$ in $\Omega \times [0, T] \Rightarrow$

$$\inf_{\Omega \times [0,T]} p \geq \inf_{\partial\Omega \times [0,T)} p$$

where $p = p_o + \widetilde{p}$.

Discrete maximum principle. Assumptions.

- zero capillary pressure: $p_c \equiv 0$
- constant porosity: $\phi \equiv const$
- incompressibility: $b_\alpha = 1, \alpha = w, o$
- $\mathbb{K}$ is strictly elliptic

- no constant viscosities assumption: $\mu_\alpha \not\equiv 0$

Discrete maximum principle. Pressure.

- Let $\mathcal{T}_{inj}$ be a set of cells where $q_w + q_o \geq 0$ and $\mathcal{T}_B$ be a set of boundary faces. Then

$$\max_{T \in \mathcal{T} \setminus (\mathcal{T}_{inj} \cup \mathcal{T}_B)} p_T \leq p_{max} = \max_{\mathcal{T}_{inj} \cup \mathcal{T}_B} p_T.$$

- Let $\mathcal{T}_{prod}$ be the set of cells where $q_w + q_o \leq 0$ and $\mathcal{T}_B$ be a set of boundary faces. Then

$$\min_{T \in \mathcal{T} \setminus (\mathcal{T}_{prod} \cup \mathcal{T}_B)} p_T \leq p_{min} = \min_{\mathcal{T}_{prod} \cup \mathcal{T}_B} p_T$$