Near-well correction method for complex wells

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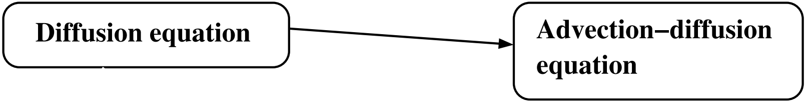
Outline

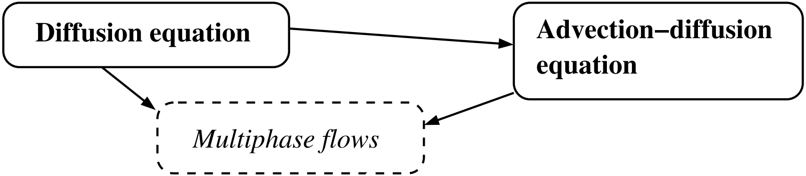
1. History & references,
2. Nonlinear FV schemes,
3. Near-well correction method,
4. Numerical experiments,
5. Complex well networks.

Diffusion equation

- C. LePotier. *Schema volumes finis monotone pour des operateurs de diffusion fortement anisotropes sur des maillages de triangle non structures.* C.C.Acad.Sci.Paris 341 (2005) 787–792.
- **2D:** K. Lipnikov, D. Svyatskiy, Y. Vassilevski. *Interpolation-free monotone finite volume method for diffusion equations on polygonal meshes.* JCP, 228(3) (2009) 703–716.
- **3D:** A. Danilov and Y. Vassilevski. *A monotone nonlinear finite volume method for diffusion equations on conformal polyhedral meshes.* RJNAMM, 24(3) (2009) 207–227.



- **2D:** K. Lipnikov, D. Svyatskiy, Y. Vassilevski. *A monotone finite volume method for advection-diffusion equations on unstructured polygonal meshes.* JCP, 229 (2010) 4017–4032.
- **3D:** K. Nikitin, Y. Vassilevski. *A monotone finite folume method for advection-diffusion equations on unstructured polyhedral meshes in 3D.* RJNAMM, 25(4) (2010) 335–358.



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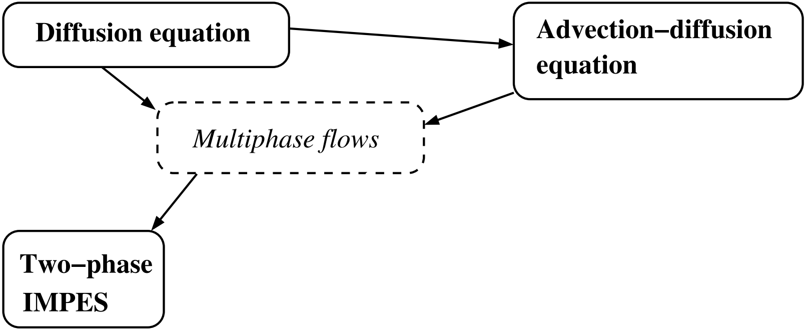

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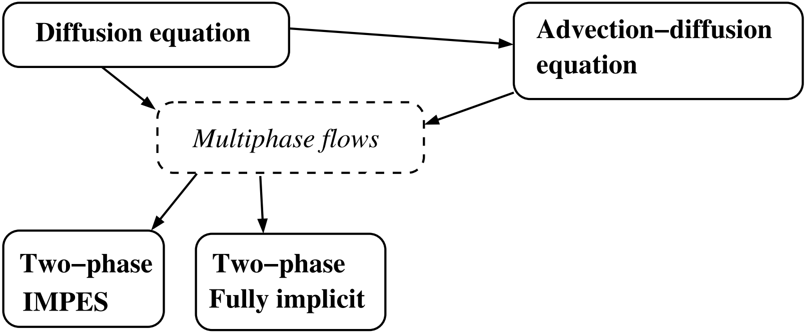

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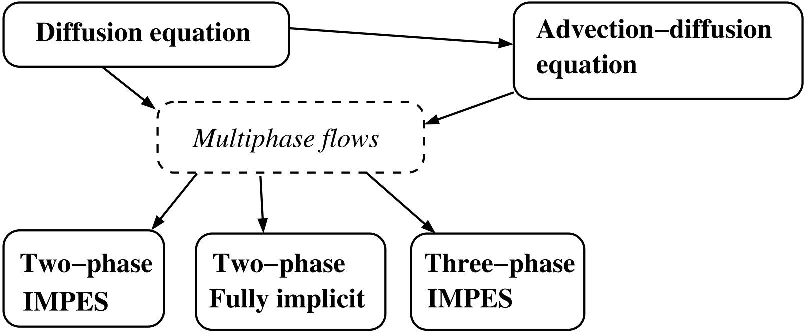

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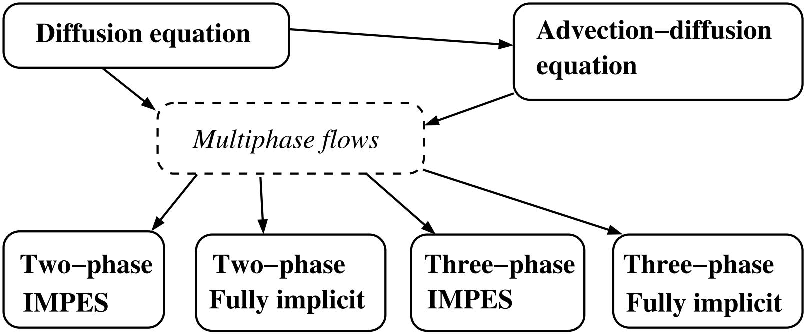

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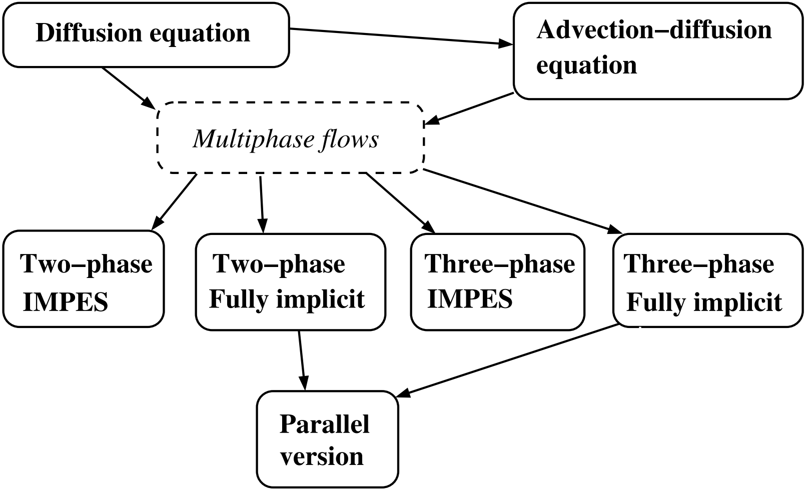

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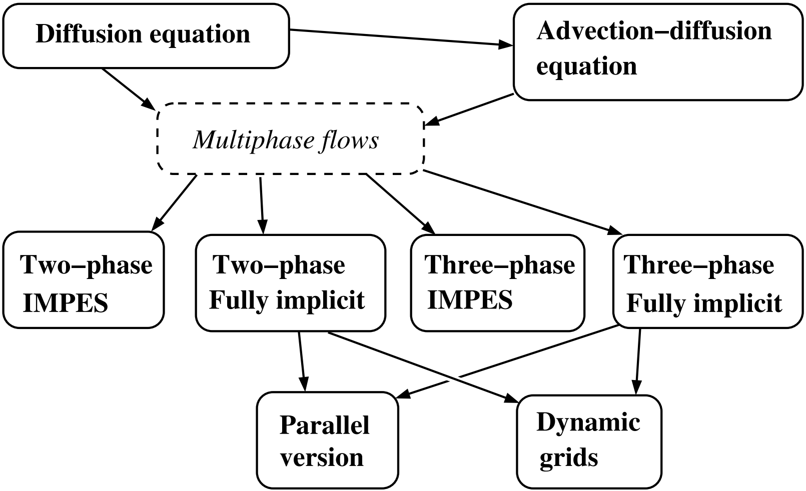

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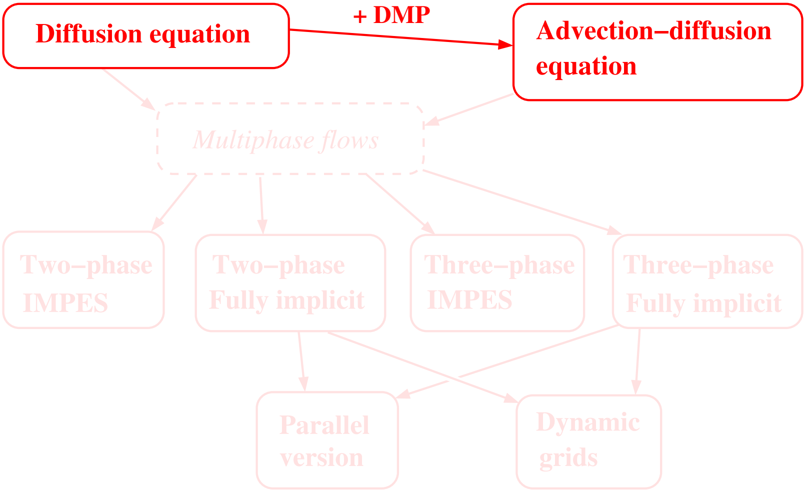

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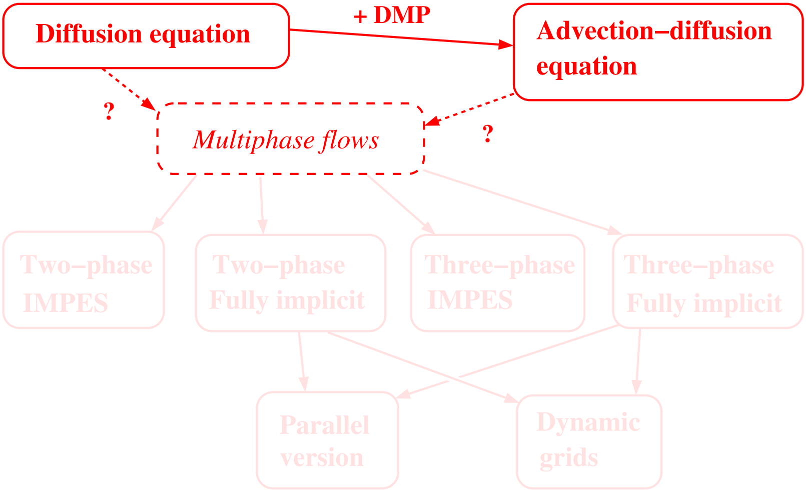

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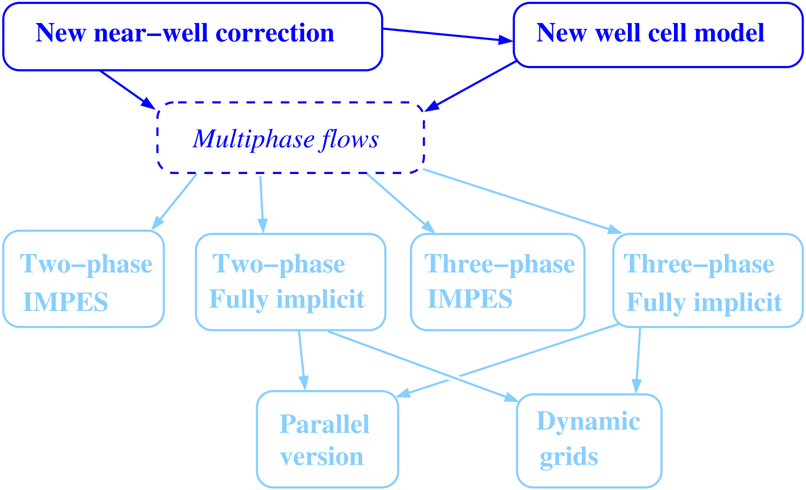

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Nonlinear FV schemes

Nonlinear scheme for diffusive flux

$$\mathbb{K}\nabla c \cdot \mathbf{n}_f =$$

$$\nabla c \cdot (\mathbb{K}\,\mathbf{n}_f) = \frac{\partial c}{\partial \boldsymbol{\ell}_f}|\boldsymbol{\ell}_f|$$

Nonlinear scheme for diffusive flux

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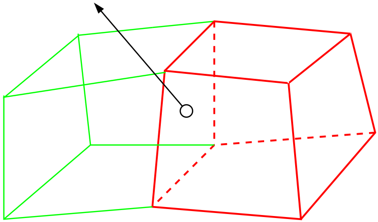

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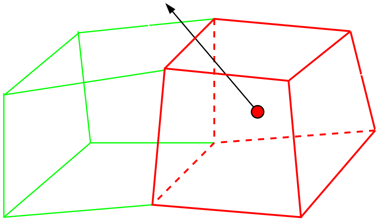

Nonlinear scheme for diffusive flux

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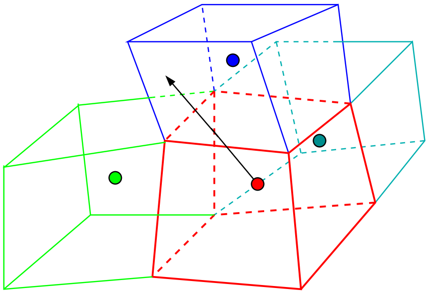
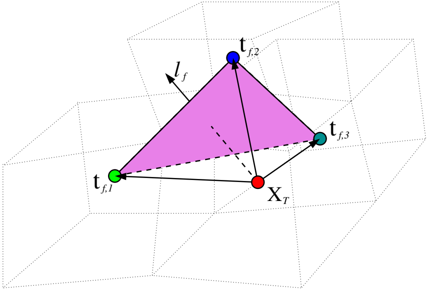


Nonlinear scheme for diffusive flux

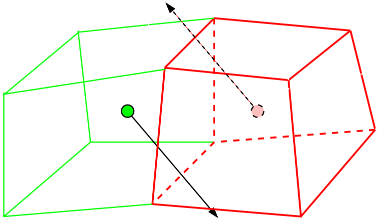
$$\mathbb{K}\nabla c \cdot \mathbf{n}_f =$$
$$\nabla c \cdot (\mathbb{K}\,\mathbf{n}_f) = \frac{\partial c}{\partial \boldsymbol{\ell}_f} |\boldsymbol{\ell}_f|$$



Nonlinear scheme for diffusive flux

$$\mathbb{K}\nabla c \cdot \mathbf{n}_f =$$

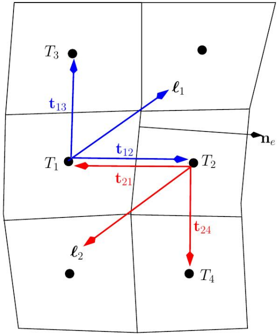
$$\nabla c \cdot (\mathbb{K}\,\mathbf{n}_f) = \frac{\partial c}{\partial \boldsymbol{\ell}_f} |\boldsymbol{\ell}_f|$$



Nonlinear scheme for diffusive flux

Monotone, two-point:

$$\begin{cases} \mu_+ + \mu_- &= 1 \\ -\mu_+ d_+ + \mu_- d_- &= 0. \\ \mu_+ \mathbf{q}_{f,+} + \mu_- \mathbf{q}_{f,-} &= 0. \end{cases}$$



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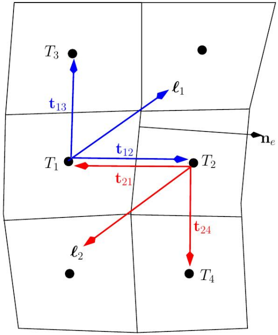
Nonlinear scheme for diffusive flux

DMP, multi-point, compact:

$$\begin{cases} \mu_+ + \mu_- & = & 1 \\ -\mu_+ d_+ + \mu_- d_- & = & 0. \\ \mu_+ \mathbf{q}_{f,+} + \mu_- \mathbf{q}_{f,-} & = & 0. \end{cases}$$



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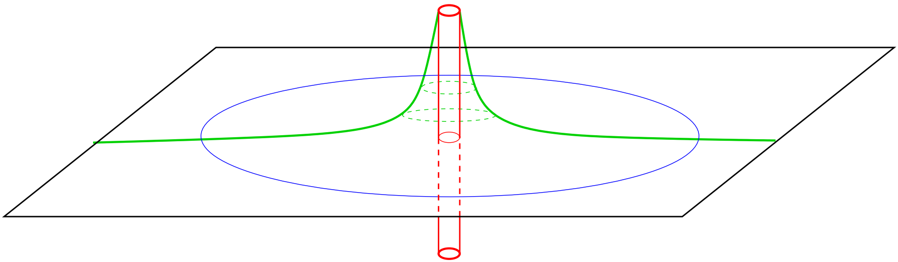
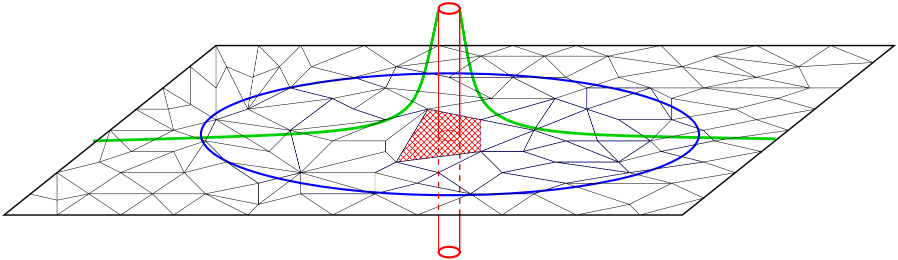
Near-well correction method

Near-well correction method



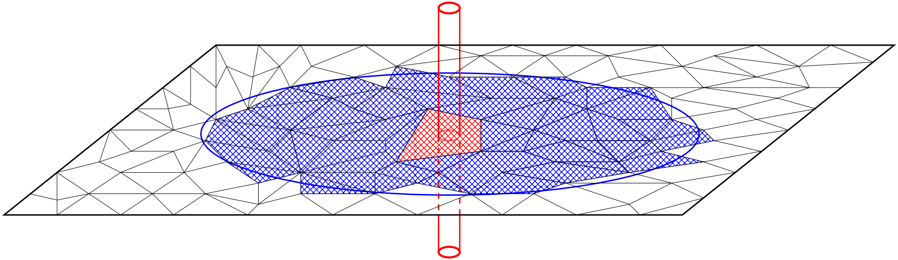
Radial flow pressure field with near-well singularity.

Near-well correction method



Unstructured grid.

Near-well correction method



Near-well region, where the NFV scheme is modified.

Near-well correction method

Assumption: Near each face *(in near-well region)* pressure is approximated by a sum of linear and nonlinear functions (e.g. logarithmic for isotropic case):

$$p_T = \underbrace{a\ x + b\ y + c\ z + d}_{p_{lin}} + \underbrace{e\ F(x, y, z)}_{p_F},$$

where $F(x, y, z)$ is a function representing the singularity.

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Near-well nonlinear correction method

Nonlinear function reconstruction:

Consider cell T_+ and neighboring cells T_i, $i = 1, 2, 3, 4$ and $P_i = P_{T_i} = a\,x_i + b\,y_i + c\,z_i + d + e\,F\,(x_i, y_i, z_i)$.

$$\begin{pmatrix} p_1 - p_+ \\ p_2 - p_+ \\ p_3 - p_+ \\ p_4 - p_+ \end{pmatrix} = \begin{pmatrix} x_1 - x_+ & y_1 - y_+ & z_1 - z_+ & F_1 - F_+ \\ x_2 - x_+ & y_2 - y_+ & z_2 - z_+ & F_2 - F_+ \\ x_3 - x_+ & y_3 - y_+ & z_3 - z_+ & F_3 - F_+ \\ x_4 - x_+ & y_4 - y_+ & z_4 - z_+ & F_4 - F_+ \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ e \end{pmatrix}.$$

Near-well nonlinear correction method

Nonlinear function reconstruction:

Consider cell T_+ and neighboring cells T_i, $i = 1, 2, 3, 4$ and $P_i = P_{T_i} = a\,x_i + b\,y_i + c\,z_i + d + e\,F\,(x_i, y_i, z_i)$.

$$\begin{pmatrix} p_1 - p_+ \\ p_2 - p_+ \\ p_3 - p_+ \\ p_4 - p_+ \end{pmatrix} = \begin{pmatrix} x_1 - x_+ & y_1 - y_+ & z_1 - z_+ & F_1 - F_+ \\ x_2 - x_+ & y_2 - y_+ & z_2 - z_+ & F_2 - F_+ \\ x_3 - x_+ & y_3 - y_+ & z_3 - z_+ & F_3 - F_+ \\ x_4 - x_+ & y_4 - y_+ & z_4 - z_+ & F_4 - F_+ \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ e \end{pmatrix}.$$

$$a_+ = \sum_j (P_j - P_+)\, m_{1,j}, \qquad b_+ = \sum_j (P_j - P_+)\, m_{2,j},$$

$$c_+ = \sum_j (P_j - P_+)\, m_{3,j}, \qquad e_+ = \sum_j (P_j - P_+)\, m_{4,j},$$

where $m_{i,j}$ is an inverse matrix elements.

Near-well nonlinear correction method

$$\mathbf{q}_{\pm} \cdot \mathbf{n}_f = \pm \Big[\ell_1 \sum_j (P_j - P_{\pm})\, m^{\pm}_{1,j} + \ell_2 \sum_j (P_j - P_{\pm})\, m^{\pm}_{2,j} + \ell_3 \sum_j (P_j - P_{\pm})\, m^{\pm}_{3,j} + \ell_4 \sum_j (P_j - P_{\pm})\, m^{\pm}_{4,j} \Big] =$$

Near-well nonlinear correction method

$$\mathbf{q}_\pm \cdot \mathbf{n}_f = \pm\Big[\ell_1 \sum_j (P_j - P_\pm)\, m^\pm_{1,j} + \ell_2 \sum_j (P_j - P_\pm)\, m^\pm_{2,j} +$$
$$\ell_3 \sum_j (P_j - P_\pm)\, m^\pm_{3,j} + \ell_4 \sum_j (P_j - P_\pm)\, m^\pm_{4,j}\Big] =$$
$$= \pm\Big[\sum_j p_j \underbrace{\sum_i \ell_i\, m^\pm_{i,j}}_{k^\pm_j} - p_\pm \sum_j \underbrace{\sum_i \ell_i\, m^\pm_{i,j}}_{k^\pm_j}\Big] = \pm\Big(\sum_j k^\pm_j\, (p_j - p_\pm)\Big).$$

Near-well nonlinear correction method

$$\mathbf{q}_\pm \cdot \mathbf{n}_f = \pm\Big[\ell_1 \sum_j (P_j - P_\pm)\, m^\pm_{1,j} + \ell_2 \sum_j (P_j - P_\pm)\, m^\pm_{2,j} + \ell_3 \sum_j (P_j - P_\pm)\, m^\pm_{3,j} + \ell_4 \sum_j (P_j - P_\pm)\, m^\pm_{4,j}\Big] =$$

$$= \pm\Big[\sum_j p_j \underbrace{\sum_i \ell_i\, m^\pm_{i,j}}_{k^\pm_j} - p_\pm \sum_j \underbrace{\sum_i \ell_i\, m^\pm_{i,j}}_{k^\pm_j}\Big] = \pm\Big(\sum_j k^\pm_j\,(p_j - p_\pm)\Big).$$

Multipoint flux approximation:

$$\mathbf{q}_f \cdot \mathbf{n}_f = \mu_+ \Big(\sum_j k^+_j\,(p_j - p_+)\Big) + \mu_- \Big(\sum_{j'} k^-_{j'} \cdot (p_{j'} - p_-)\Big).$$

Near-well nonlinear correction method

$$\mathbf{q}_{\pm} \cdot \mathbf{n}_f = \pm\Big[\ell_1 \sum_j (P_j - P_{\pm})\, m^{\pm}_{1,j} + \ell_2 \sum_j (P_j - P_{\pm})\, m^{\pm}_{2,j} + \ell_3 \sum_j (P_j - P_{\pm})\, m^{\pm}_{3,j} + \ell_4 \sum_j (P_j - P_{\pm})\, m^{\pm}_{4,j}\Big] =$$

$$= \pm\Big[\sum_j p_j \underbrace{\sum_i \ell_i\, m^{\pm}_{i,j}}_{k_j^{\pm}} - p_{\pm} \sum_j \underbrace{\sum_i \ell_i\, m^{\pm}_{i,j}}_{k_j^{\pm}}\Big] = \pm\Big(\sum_j k_j^{\pm}\,(p_j - p_{\pm})\Big).$$

Multipoint flux approximation (e.g. $\mu_+ = \mu_- = 0.5$):

$$\mathbf{q}_f \cdot \mathbf{n}_f = \mu_+ \Big(\sum_j k_j^{+}\,(p_j - p_+)\Big) + \mu_- \Big(\sum_{j'} k_{j'}^{-} \cdot (p_{j'} - p_-)\Big).$$

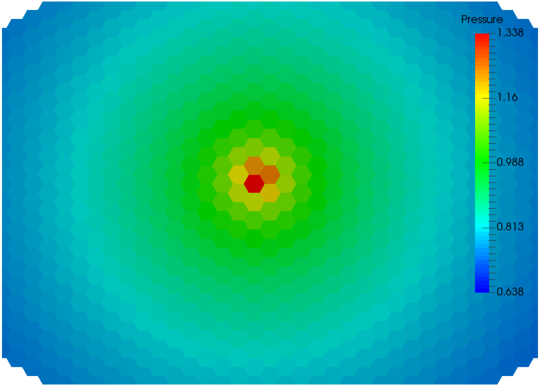
Near-well nonlinear correction method

Note: We construct the *linear* multipoint flux discretization. One may also construct a *nonlinear* scheme using pressure-dependant coefficients.

Numerical experiments

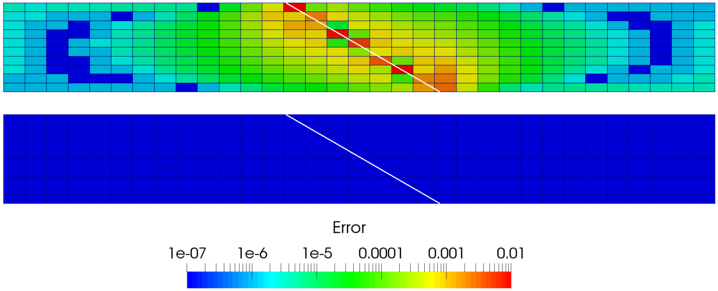
Isolated well

Shifted well:



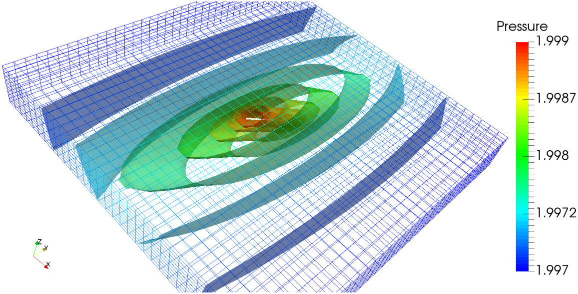
Isolated well

Slanted well + isotropic media.
Error:



Isolated well

Slanted well + anisotropic media.
Solution:



Isolated well

Slanted well + anisotropic media.
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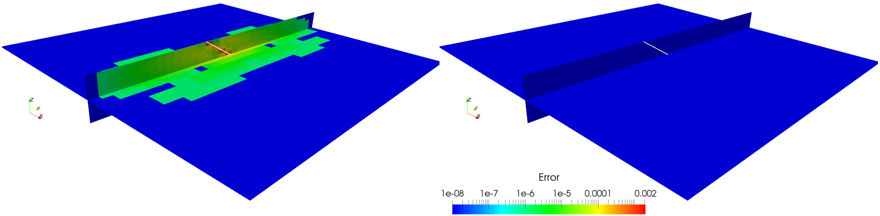
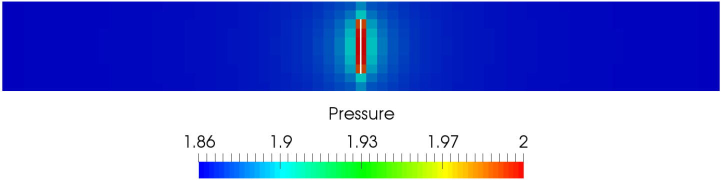


Isolated well

Partially perforated well.
Solution:



Reference solution:

V. Kramarenko, K. Nikitin, Y. Vassilevski. *A finite volume scheme with improved well modeling in subsurface flow simulation*. Computational Geosciences, Vol.21, No.5-6, (2017), pp.1023–1033.

Isolated well

Partially perforated well.
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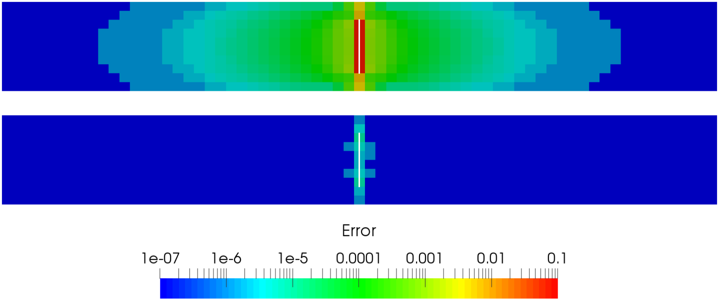
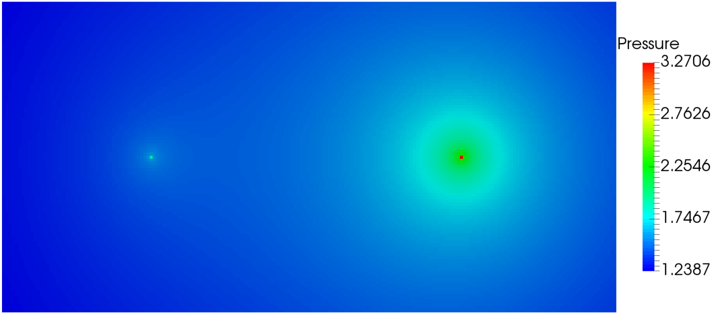


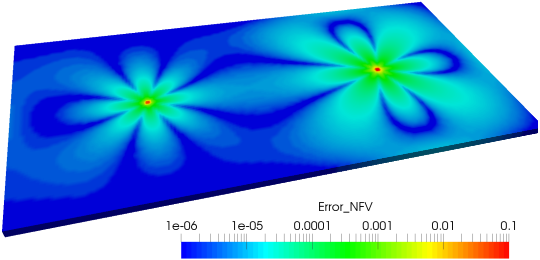
Multiple wells

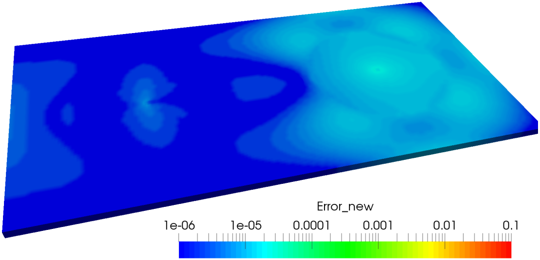
Analytical solution:



$$p = \frac{q_1 \cdot \log r_1}{2\pi k h_w} + \frac{q_2 \cdot \log r_2}{2\pi k h_w} + C.$$

Multiple wells





Multiple wells

$100/h$	$\varepsilon^{NFV}_{p,anl}$	$\varepsilon^{NWC}_{p,anl}$	$\varepsilon^{NFV}_{p,pcm}$	ε^{NWC}_{p}
33	1.2e-2	2.8e-5	1.2e-2	2.8e-5
67	5.1e-3	7.0e-6	5.2e-3	7.6e-6
99	3.1e-3	3.2e-6	3.1e-3	4.1e-6

$100/h$	$\varepsilon^{NFV}_{q_1}$	$\varepsilon^{NFV}_{q_2}$	$\varepsilon^{NWC}_{q_1}$	$\varepsilon^{NWC}_{q_2}$
33	4.6e-3	1.9e-2	2.1e-5	4.1e-5
67	4.6e-3	1.9e-2	2.3e-5	5.4e-5
99	4.6e-3	1.8e-2	2.0e-5	7.0e-5

Multiple wells

$100/h$	$\varepsilon^{NFV}_{p,anl}$	$\varepsilon^{NWC}_{p,anl}$	$\varepsilon^{NFV}_{p,pcm}$	ε^{NWC}_{p}
33	1.2e-2	2.8e-5	1.2e-2	2.8e-5
67	5.1e-3	7.0e-6	5.2e-3	7.6e-6
99	3.1e-3	3.2e-6	3.1e-3	4.1e-6

$100/h$	$\varepsilon^{NFV}_{q_1}$	$\varepsilon^{NFV}_{q_2}$	$\varepsilon^{NWC}_{q_1}$	$\varepsilon^{NWC}_{q_2}$
33	4.6e-3	1.9e-2	2.1e-5	4.1e-5
67	4.6e-3	1.9e-2	2.3e-5	5.4e-5
99	4.6e-3	1.8e-2	2.0e-5	7.0e-5

Multiple wells

$100/h$	$\varepsilon_{p,anl}^{NFV}$	$\varepsilon_{p,anl}^{NWC}$	$\varepsilon_{p,pcm}^{NFV}$	ε_{p}^{NWC}
33	1.2e-2	2.8e-5	1.2e-2	2.8e-5
67	5.1e-3	7.0e-6	5.2e-3	7.6e-6
99	3.1e-3	3.2e-6	3.1e-3	4.1e-6

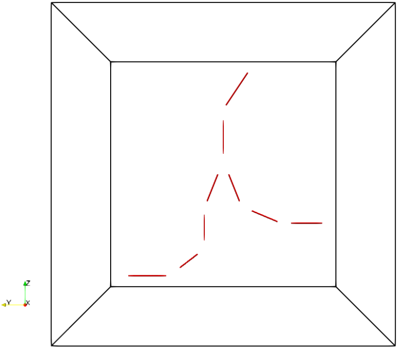
$100/h$	$\varepsilon_{q_1}^{NFV}$	$\varepsilon_{q_2}^{NFV}$	$\varepsilon_{q_1}^{NWC}$	$\varepsilon_{q_2}^{NWC}$
33	4.6e-3	1.9e-2	2.1e-5	4.1e-5
67	4.6e-3	1.9e-2	2.3e-5	5.4e-5
99	4.6e-3	1.8e-2	2.0e-5	7.0e-5

R	49	45	40	30	20	10
ε_{p}^{NWC}	6.8e-6	6.9e-6	7.1e-6	7.6e-6	9.1e-6	1.5e-5

Complex well networks

Complex well networks: segmentation

- Well network
- Segmentation
- Different p_F for each segment
- Local cylindrical grids
- Local grid for each well segment
- Interpolation of the local grid solution



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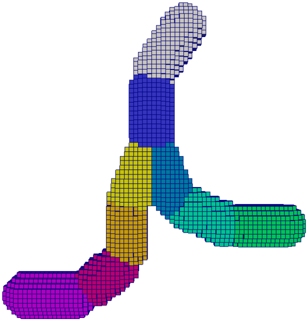

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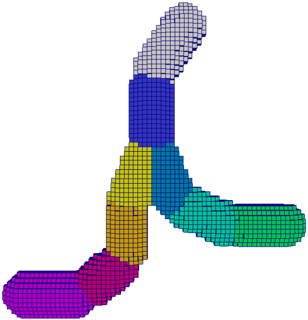

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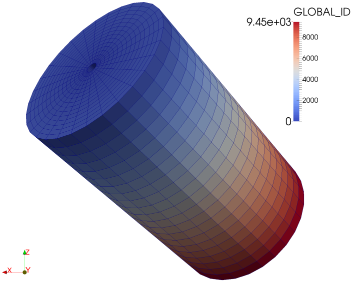

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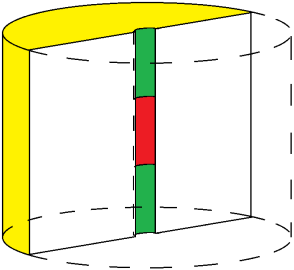

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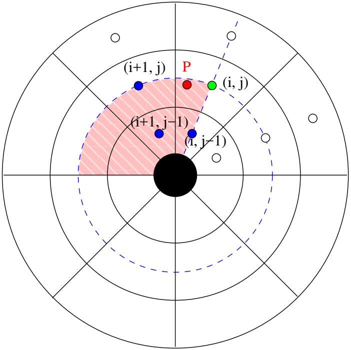

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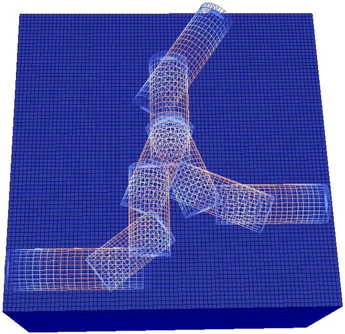

Complex well networks: segmentation

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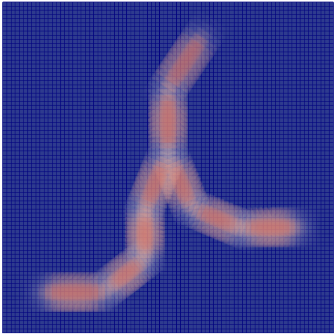


Summary

Summary

The new near-well correction scheme was suggested.

- It replaces the conventional Peaceman well model and the original FV scheme in the near-well region;
- It is applicable to the cases of arbitrary polyhedral cells, slanted wells and wells shifted from grid cell centers;
- It allows to increase the accuracy of the well-driven flows computation in the near-well region for single or multiple wells;
- Reducing the near-well region radius still provides accurate results since the major error is located in the nearest to well grid cells;
- It can be extended to the case of the complex well networks.

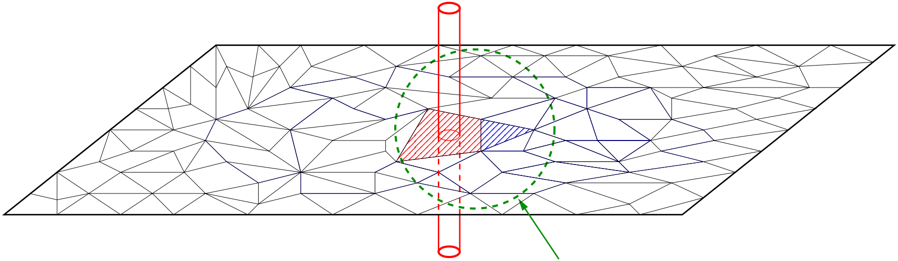
Thank you for your attention!

http://www.inmost.org

http://dodo.inm.ras.ru/research/fvmon

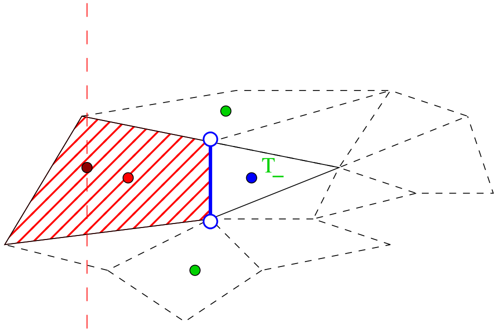
Appendix

Well cell model

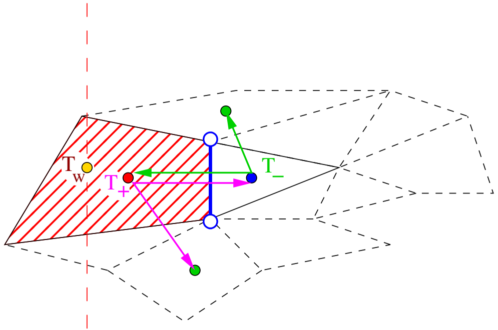


Consider well cell and the face shared with one of the neighbors.

Well cell model

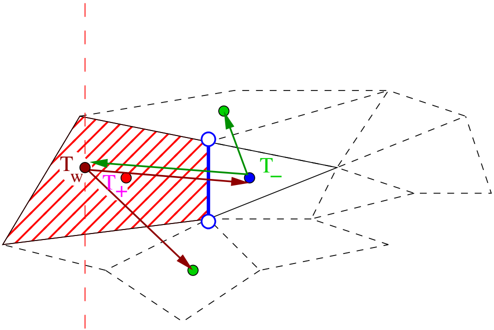


Well cell model



$$\mathbf{q}_f \cdot \mathbf{n}_f = \mu_+ \left(\sum_j k_j^+ \left(p_j - p_+\right)\right) + \mu_- \left(\sum_{j'} k_{j'}^- \left(p_{j'} - p_-\right)\right).$$

Well cell model



$$\mathbf{q}_f \cdot \mathbf{n}_f = \mu_+ \left(\sum_l k_l^+ \left(p_l - p_w \right) \right) + \mu_- \left(\sum_{l'} k_{l'}^- \left(p_{l'} - p_- \right) \right).$$

Well cell model

For given *bottom hole pressure* p_w:

$$\sum_f \left[\frac{1}{2} \Big(\sum_j k_j^+ \, (p_j - p_+) \Big) - \frac{1}{2} \Big(\sum_{j'} k_{j'}^- \, (p_{j'} - p_-) \Big) \right] = \sum_f \left[1 \left(\sum_l k_l^+ \, (p_l - p_w) \right) - 0 \left(\ldots \right) \right].$$

Well cell model

For given *well rate* q_w:

$$\sum_f \left[\frac{1}{2} \Big(\sum_j k_j^+ \, (p_j - p_+) \Big) - \frac{1}{2} \Big(\sum_{j'} k_{j'}^- \, (p_{j'} - p_-) \Big) \right] = \sum_f \left[1 \left(\sum_l k_l^+ \, (p_l - p_w) \right) - 0 \left(\ldots \right) \right].$$

With additional equation for unknown p_w:

$$\sum_{\text{well cells}} \sum_f \left(\sum_l k_l^+ \, (p_l - p_w) \right) = q_w.$$