New Homogenization Method for Diffusion Problems: Numerical Results

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- Classical problem formulation
- Saddle point formulation
- Asymptotic expansions
- Convergence estimate
- Numerical results 1
- Numerical results 2

Let Ω be a polygon and ω_s be non-overlapping polygonal sub-domains in Ω, such that $\partial\omega_s \cap \partial\omega_t = \varnothing$ and $\partial\omega_s \cap \partial\Omega = \varnothing$, $s,t = \overline{1,m}$.

We consider the diffusion problem

$$\begin{array}{rcll} -\nabla\left[K(x)\nabla u\right] & = & f & \text{in} \quad \Omega, \\ u & = & 0 & \text{on} \quad \partial\Omega \end{array} \tag{1}$$

with the diffusion coefficient

$$K(x) = \begin{cases} 1 + \frac{1}{\varepsilon_s} & \text{in} \quad \omega_s, \ s = \overline{1,m}, \\ 1 & \text{in} \quad \Omega \setminus \overline{\omega}_s, \end{cases} \tag{2}$$

where $\varepsilon_s \equiv \text{const} > 0$ in ω_s, $s = \overline{1,m}$.
We shall call $\{\omega_s\}$ inclusions.

Variational Problem Formulation

In the variational form: find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega (K(x)\nabla u)\cdot\nabla v\,dx \equiv \int_\Omega \nabla u\cdot\nabla v\,dx + \sum_{s=1}^{m} \frac{1}{\varepsilon_s}\int_{\omega_s} \nabla u\cdot\nabla v\,dx = \int_\Omega f v\,dx \tag{3}$$

$\forall v \in H_0^1(\Omega)$

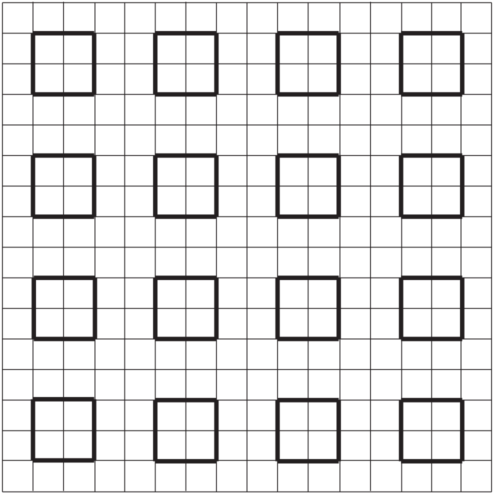

Figure: Periodic distribution of ω_s, $s = \overline{1, m}$ with $d = \frac{1}{8}$, $h = \frac{1}{16}$, $m = 16$.

Example of Ω

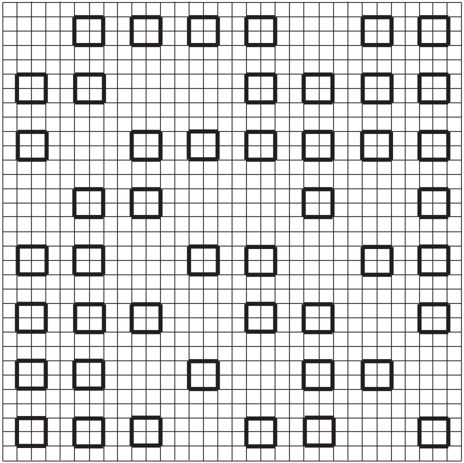

Figure: Random distribution of ω_s, $s = \overline{1, m}$ with $d = \frac{1}{16}$, $h = \frac{1}{32}$, $m = 46$.

Example of Ω

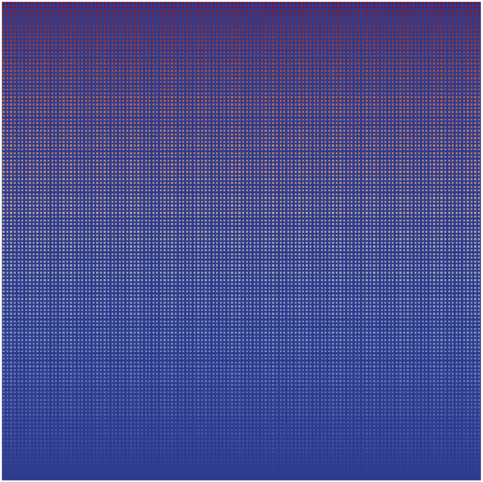

Figure: Large number of inclusions.

FEM problem

Let Ω_h be a triangular mesh in Ω conforming with $\partial\Omega$ and $\partial\omega_s$, $s = \overline{1,m}$, and V_h be the P1 finite element subspace of $H_0^1(\Omega)$. Then the finite element discretization to (3) reads as follows: find $u_{\varepsilon,h} \in V_h$, such that

$$\int_\Omega \nabla u_{\varepsilon,h} \cdot \nabla v_h \, dx + \sum_{s=1}^{m} \frac{1}{\varepsilon_s} \int_{\omega_s} \nabla u_{\varepsilon,h} \cdot \nabla_h \, dx = \int_\Omega f v_h \, dx \qquad (4)$$

The above P1 FEM results in the algebraic system

$$A_\varepsilon \overline{u}_\varepsilon \equiv A\overline{u}_\varepsilon + \begin{bmatrix} \Sigma_s^{-1} B_\omega & 0 \\ 0 & 0 \end{bmatrix} \overline{u}_\varepsilon = \overline{f} \qquad (5)$$

with the matrix $A \sim -\triangle_h$:

$$(A\overline{u}, \overline{v}) = \int\limits_{\Omega} \nabla u_h \cdot \nabla v_h \, dx \ \forall u_h, v_h \in V_h, \tag{6}$$

with the block diagonal matrix

$$B_\omega = \text{diag}\Big\{B_{\omega,1}, \ldots, B_{\omega,m}\Big\} \tag{7}$$

where

$$(B_{\omega,s}, \overline{u}_s, \overline{v}_s) = \int\limits_{\omega_s} \nabla u_{s,h} \cdot \nabla v_{s,h} \, dx \ \forall u_{s,h}, v_{s,h} \in V_{s,h}, \ \ s = \overline{1, m}, \tag{8}$$

and with the block diagonal matrix

$$\Sigma_s = \text{diag}\Big\{\varepsilon_1 I_1, \ldots, \varepsilon_m I_m\Big\} \tag{9}$$

where I_s are the underlying identity matrices, $s = \overline{1, m}$. Here, $V_{s,h}$ is the restriction of V_h onto $\overline{\omega}_s$, $s = \overline{1, m}$.

Equivalent saddle-point system

Define sub-vectors

$$\overline{p}_{\varepsilon,s} = \frac{1}{\varepsilon}\overline{u}_{\varepsilon,s} \tag{10}$$

$s = \overline{1,m}$. We replace the original algebraic system (5) with the s.p.d. matrix A_ε by the equivalent saddle-point system

$$\begin{array}{cccccc} A\overline{u}_\varepsilon & + & B^T\overline{p}_\varepsilon & = & \overline{f} \\ B\overline{u}_\varepsilon & - & \Sigma_\varepsilon B_\omega \overline{p}_\varepsilon & = & 0 \end{array} \tag{11}$$

where

$$B = \begin{bmatrix} B_\omega & 0 \end{bmatrix} \tag{12}$$

and

$$\overline{p}_\varepsilon^T = \left[\overline{p}_{\varepsilon,1}^T, \ldots, \overline{p}_{\varepsilon,m}^T\right] \tag{13}$$

Replace the original FE problem by the problem: find $u_{\varepsilon,h} \in V_h$, $p_{\varepsilon,s,h} \in V_{s,h}$, $s = \overline{1,m}$ such that

$$\int\limits_{\Omega_h} (\nabla u_{\varepsilon,h}) \cdot \nabla v_h \, dx + \sum_{s=1}^{m} \int\limits_{\omega_s} (\nabla p_{\varepsilon,s,h}) \cdot \nabla v_h \, dx = \int\limits_{\Omega} f v_h \, dx \quad (14)$$

$$\int\limits_{\omega_s} (\nabla u_{\varepsilon,s}) \cdot q_{s,h} \, dx - \varepsilon_s \int\limits_{\omega_s} (\nabla p_{\varepsilon,s,h}) \cdot \nabla q_{s,h} \, dx = 0 \quad (15)$$

$\forall v_h \in V_h$, $q_{s,h} \in V_{s,h}$, $s = \overline{1,m}$. This FE problem results in the same saddle-point algebraic system (11).
Here

$$p_{\varepsilon,s,h} = \frac{1}{\varepsilon_s} u_{\varepsilon,s} \text{ in } \overline{\omega}_s, \;\; s = \overline{1,m} \quad (16)$$

Historical remark

The above replacement of system (5) by system (11) as well as the preconditioned iterative solution methods for system (11) were proposed in [Kuznetsov 2000] for the case of one inclusion and in [Kuznetsov 2009] for the case of multiple inclusions.

Yu. Kuznetsov, "New Iterative Methods for Singular Perturbed Positive Definite Matrices", *Russian Journal of Numerical Analysis and Mathematical Modeling*, **15:1**, 2000, pp. 65–71.

Yu. Kuznetsov, "Preconditioned iterative methods for algebraic saddle-point problems", *Russian Journal of Numerical Mathematics*, **17:1**, 2009, pp. 67–75.

Homogenization method

Assume that $\varepsilon_s = \varepsilon$, $s = \overline{1, m}$. For approximation of the solution of the system

$$\mathcal{A}_\varepsilon \begin{bmatrix} \overline{u}_\varepsilon \\ \overline{p}_\varepsilon \end{bmatrix} \equiv \begin{bmatrix} A & B^T \\ B & -\Sigma B_\omega \end{bmatrix} \begin{bmatrix} \overline{u}_\varepsilon \\ \overline{p}_\varepsilon \end{bmatrix} = \begin{bmatrix} \overline{f} \\ 0 \end{bmatrix} \tag{17}$$

we apply the "homogenization" method based on the expansions

$$\overline{u}_\varepsilon^{(r)} = \sum_{l=0}^{r} \varepsilon^l \, \overline{u}^{(l)}, \tag{18}$$

$$\overline{p}_\varepsilon^{(r)} = \sum_{l=0}^{r} \varepsilon^l \, \overline{p}^{(l)}, \tag{19}$$

where the vectors $\overline{u}^{(l)}$, $\overline{p}^{(l)}$, $l = \overline{0, r}$ satisfy the systems

$$\mathcal{A}_0 \begin{bmatrix} \overline{u}^{(0)} \\ \overline{p}^{(0)} \end{bmatrix} \equiv \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \overline{u}^{(0)} \\ \overline{p}^{(0)} \end{bmatrix} = \begin{bmatrix} \overline{f} \\ 0 \end{bmatrix} \tag{20}$$

and

$$\mathcal{A}_0 \begin{bmatrix} \overline{u}^{(l)} \\ \overline{p}^{(l)} \end{bmatrix} = \begin{bmatrix} 0 \\ B_\omega \overline{p}^{(l-1)} \end{bmatrix} \tag{21}$$

Remark: Without loss of generality we may assume that the solution vectors $\overline{p}_\varepsilon$ in (19) and $\overline{p}^{(l)}$ in (20), (21) are orthogonal to $\ker B_\omega$. We recall that $\ker B_{\omega,s}$ consisits of the vectors with constant components, $s = \overline{1, m}$. To this end $\dim \ker B_\omega = m$.

The error estimate

$$\|\overline{u}_\varepsilon - \overline{u}_\varepsilon^{(r)}\|_A \leqslant \|\overline{p}_\varepsilon - \overline{p}_\varepsilon^{(r)}\|_{B_\omega} \leqslant \frac{\varepsilon^r}{\mu_{h,*}^{r+1}}\|f\|_{A^{-1}} \tag{22}$$

can be derived [Kuznetsov 2018]. It was proved that in the case of regular shaped quasiuniform meshes Ω_h the value of $\mu_{h,*}$ is bounded from below by a positive constant independent of Ω_h. where $\mu_{h,*}$ is the minimal nonzero eigenvalue of the eigenvalue problem

$$BA^{-1}B^T\overline{w} = \mu B_\omega \overline{w} \tag{23}$$

Yu. Kuznetsov, "New homogenization method for diffusion equations", *Russian Journal of Numerical Mathematics*, **33:2**, 2018, pp. 85–93.

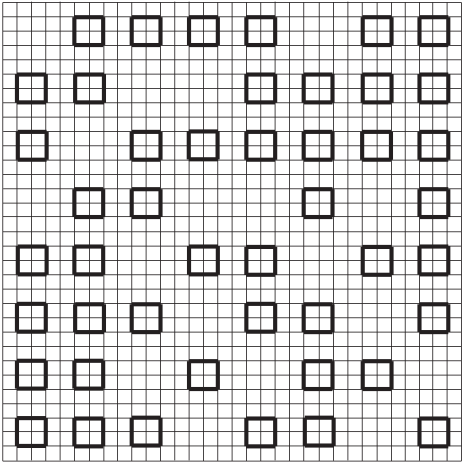

Figure: Random distribution of ω_s, $s = \overline{1, m}$ with $d = \frac{1}{16}$, $h = \frac{1}{32}$, $m = 46$.

Let $\Omega = (0:1) \times (0:1)$ be the unite square with square $d \times d$ inclusions ω_s, $s = \overline{1, m}$, and Ω_h be a square mesh with the mesh step size $h = \frac{d}{2^t}$, $t = 1, 2 \ldots$. In numerical test we compute the values

$$\delta \overline{u}^{(l)} = \frac{\|\overline{u}_\varepsilon - \overline{u}^{(r)}\|_A}{\|f\|_{A^{-1}}} \qquad \delta \overline{p}^{(l)} = \frac{|||\overline{p}_\varepsilon - \overline{p}^{(r)}|||_{B_\omega}}{\|f\|_{A^{-1}}}, \quad r = 0, 1, 2, \ldots \tag{24}$$

Here,

$$\|\overline{\psi}_A\| = \left(A\overline{\psi}, \overline{\psi}\right)^{\frac{1}{2}} \qquad |||\overline{q}|||_{B_\omega} = (B_\omega \overline{q}, \overline{q})^{\frac{1}{2}} \tag{25}$$

To compute the vectors $\overline{u}_\varepsilon$, $\overline{p}_\varepsilon$, $\overline{u}^{(r)}$, $\overline{p}^{(r)}$, $r = 0, 1, 2, \ldots$ we use the PCG Uzawa with the preconditioner B_ω^+, i.e. we solve the systems

$$\left(\Sigma_\varepsilon B_\omega + BA^{-1}B^T\right)\overline{p} = \overline{g}, \quad g \in \mathrm{im} B_\omega \tag{26}$$

by the classical PCG method.

In periodic distribution of $\{\omega_s\}$ the distances between neighboring ω_s and ω_t, $s \neq t$ are equal to d, whereas distance between ω_s and Ω is equal to $d/2$, $s, t = \overline{1, m}$. Random distribution of $\{\omega_s\}$ is a subset of the periodic distribution. We solve the systems with the matrices A in the Uzawa algorithm by the PCG method with the AMG preconditioner. It takes about 12-14 AMG-PCG iterations to solve a system with relative accurace 10^{-8}

Remark: To implement the preconditioner B_ω^+ - generalized inverse of B_ω we use the fact that

$$B_\omega^+ B = B_\omega^+ \begin{bmatrix} B_\omega & 0 \end{bmatrix} = \begin{bmatrix} I - Q & 0 \end{bmatrix} \tag{27}$$

where Q is the M-orthogonal projector onto $\ker B_\omega$ where M is the underlying mass matrix.

Example of inclusion distribution

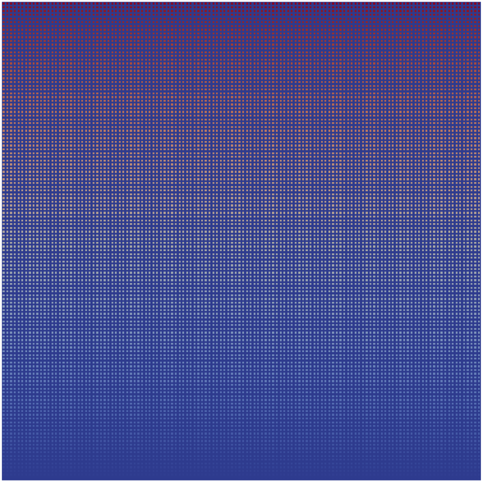

Figure: 16384 inclusions.

Model Problem 1:numerical results

ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$\delta u^{(0)}$	9.3e-2	8.0e-3	1.0e-3	1.0e-4
$\delta p^{(0)}$	1.6e-1	1.9e-2	2.0e-3	2.0e-4
$\delta u^{(1)}$	2.3e-2	2.6e-4	2.9e-6	2.9e-8
$\delta p^{(1)}$	4.0e-2	5.4e-4	5.0e-5	5.0e-8
$\delta u^{(2)}$	6.6e-3	9.5e-6	8.3e-9	8.4e-12
$\delta p^{(2)}$	1.0e-2	1.6-e5	1.3e-8	1.2e-11
$\delta u^{(3)}$	1.8e-3	3.0e-7	2.3e-11	6.5e-13
$\delta p^{(3)}$	2.6e-3	5.3e-7	3.4e-11	8.5e-13

Table: $d = \frac{1}{256}$, $h = \frac{1}{512}$, $m = 16384$

ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$\delta u^{(0)}$	7.3e-2	9.6e-3	9.0e-4	9,8e-5
$\delta p^{(0)}$	1.6e-1	2.0e-2	2.0e-3	2.0e-4
$\delta u^{(1)}$	2.2e-2	2.9e-4	3.0e-6	3.0e-8
$\delta p^{(1)}$	4.4e-2	5.4e-4	5.5e-6	5.5e-8
$\delta u^{(2)}$	7.0e-3	9.1e-6	9.4e-9	9,2e-12
$\delta p^{(2)}$	1.2e-2	1.5e-5	1.6e-8	1.5e-11
$\delta u^{(3)}$	2.2e-3	2.8e-7	3.1e-11	5.9e-13
$\delta p^{(3)}$	3.7e-3	4.6e-7	5.0e-11	6.4e-13

Table: $d = \frac{1}{128}$, $h = \frac{1}{512}$, 4096

Model Problem 1:numerical results

ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$\delta u^{(0)}$	6.3e-2	8.0e-3	8.0e-4	8.3e-5
$\delta p^{(0)}$	1.6e-1	1.9e-2	2.0e-3	2.0e-4
$\delta u^{(1)}$	2.0e-3	2.6e-4	2.7e-6	2.7e-8
$\delta p^{(1)}$	4.3e-2	5.4e-4	5.6e-6	5.6e-8
$\delta u^{(2)}$	6.9e-3	9.0e-6	9.2e-9	4.1e-11
$\delta p^{(2)}$	1.3e-4	1.6e-5	1.7e-8	5.0e-11
$\delta u^{(3)}$	2.3e-3	3.0e-7	4.4e-11	4.0e-11
$\delta p^{(3)}$	4.1e-3	5.3e-7	7.5e-11	4.8e-11

Table: $d = \frac{1}{64}$, $h = \frac{1}{512}$, $m = 1024$

Model problem 2:numerical results

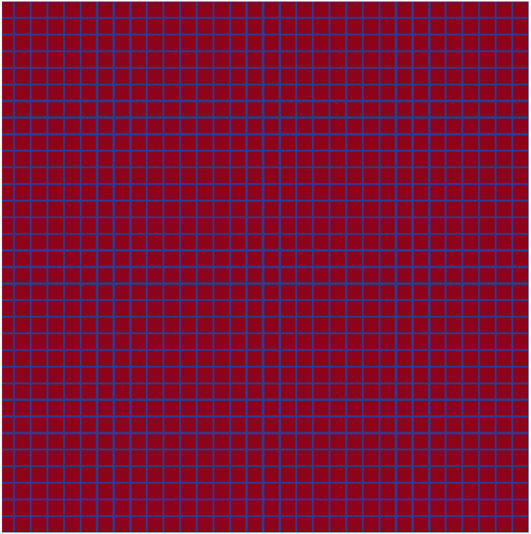

Figure: Regular distribution of ω_s, $s = \overline{1, m}$ with ratio 1:7.

Model problem 2:numerical results

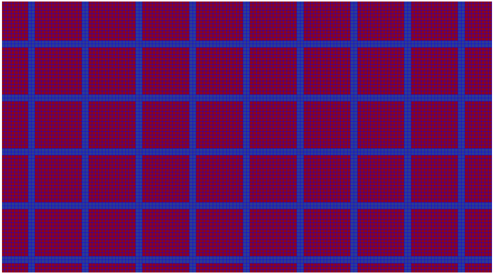

Figure: Regular distribution of ω_s, $s = \overline{1, m}$ with ratio 1:7.

In this set of numerical experiments we investigate dependence of $\mu_{h,*}$ of the ratio $d_{st} : d_s$.
We also investigate the accuracy of the approximate solutions of the value

$$q_{\varepsilon,h} = \frac{\varepsilon}{\mu_{h,*}}. \tag{28}$$

Model problem 2:numerical results

ratio $d_s : d_{st}$	$\mu_{h,*}$	Number of Uzawa-PCG iterations
1:1 (m=16384)	0.216	11
1:3 (m=4096)	0.135	14
1:7 (m=1024)	0.077	17
1:15 (m=256)	0.042	21
1:31 (m=64)	0.021	23
1:63 (m=8)	0.011	25

Table: $h = \frac{1}{512}$ periodic distribution.

In this series of numerical experiments we consider the situation when distances $d_{s,t}$ between periodically distributed inclusions are much smaller then the series d_s of the inclusions. We compute the underlying values of $\mu_{h,*}$ and compare the theoretical estimates [Kuznetsov 2018]

$$\frac{\|\overline{u}_\varepsilon - \overline{u}^{(r)}\|_A}{\|f\|_{A^{-1}}} \leqslant \delta\overline{p}^{(l)} = \frac{\|\overline{p}_\varepsilon - \overline{p}^{(r)}\|_{B_\omega}}{\|f\|_{A^{-1}}} \leqslant \frac{\varepsilon^r}{\mu_{h,*}^{r+1}} \tag{29}$$

with numerically obtained results. We also give the numbers of Uzawa PCG iterations needed to solve the underlying saddle point algebraic systems with the relative accuracy 10^{-6}.

Model problem 2:numerical results

ratio	1:1	1:3	1:7	1:15	1:31
$q_{\varepsilon,h} = \frac{\varepsilon}{\mu_{h,*}}$	0.462	0.740	1.29	2.38	4.76
$\delta u^{(0)}$	8.7e-2	5.6e-2	4.0e-2	3.3e-2	3.8e-2
$\delta p^{(0)}$	1.6e-1	1.3e-1	1.1e-1	1.0e-1	1.3e-1
$\delta p^{(1)}$	2.3e-2	1.6e-2	1.4e-2	3.2e-2	1.0e-1
$\delta u^{(1)}$	4.0e-2	3.0e-2	2.9e-2	7.3e-2	3.3e-1
$\delta u^{(2)}$	6.5e-3	5.7e-3	9.3e-3	5.4e-2	3.6e-1
$\delta p^{(2)}$	1.0e-2	9.4-e3	1.7e-2	1.2e-1	1.15
$\delta u^{(3)}$	1.8e-3	2.0e-3	7.7e-3	9.6e-2	1.27
$\delta p^{(3)}$	2.6e-3	3.2e-3	1.5e-2	2.3e-1	4.06

Table: $\varepsilon = 10^{-1}$,$h = \frac{1}{512}$

Model problem 2:numerical results

ratio	1:1	1:3	1:7	1:15	1:31
$q_{\varepsilon,h} = \frac{\varepsilon}{\mu_{h,*}}$	0.0462	00740	0.129	0.238	0.476
$\delta u^{(0)}$	2.0e-2	7.0e-3	5.0e-3	4.8e-3	9.6e-3
$\delta p^{(0)}$	1.0e-2	1.6e-2	1.3e-2	1.4e-2	3.5e-2
$\delta p^{(1)}$	2.8e-4	2.1e-4	2.1e-4	7.1e-4	3.2e-3
$\delta u^{(1)}$	4.9e-4	3.8e-4	4.2e-2	1.6e-3	1.1e-2
$\delta u^{(2)}$	8.1e-6	7.5e-6	1.5e-5	1.2e-4	1.1e-3
$\delta p^{(2)}$	1.2e-5	1.2-e5	3.0e-5	3.0e-4	4.1e-3
$\delta u^{(3)}$	2.2e-7	2.7e-7	1.3e-6	2.2e-5	4.4e-4
$\delta p^{(3)}$	3.3e-7	4.3e-7	2.7e-6	5.8e-5	1.4e-3

Table: $\varepsilon = 10^{-2}$,$h = \frac{1}{512}$

Model problem 2:numerical results

ratio	1:1	1:3	1:7	1:15	1:31
$q_{\varepsilon,h} = \frac{\varepsilon}{\mu_{h,*}}$	0.00462	0.00740	0.0129	0.0238	0.0476
$\delta u^{(0)}$	1.0e-3	7.0e-4	5.1e-4	5.3e-4	1.2e-4
$\delta p^{(0)}$	2.0e-3	1.6e-3	1.3e-3	1.5e-3	4.3e-4
$\delta p^{(1)}$	2.9e-6	2.2e-6	2.3e-6	7.9e-6	4.4e-5
$\delta u^{(1)}$	5.0e-6	3.9e-6	4.5e-6	1.9e-5	1.4e-4
$\delta u^{(2)}$	8.3e-9	7.7e-9	1.6e-8	1.3e-7	1.7e-6
$\delta p^{(2)}$	1.2e-8	1.2e-8	3.2e-8	3.5e-7	5.9e-6
$\delta u^{(3)}$	2.3e-11	5.2e-11	1.4e-10	1.0e-8	1.0e-7
$\delta p^{(3)}$	3.2e-11	7.3e-11	3.0e-10	2.1e-8	2.9e-7

Table: $\varepsilon = 10^{-3}$,$h = \frac{1}{512}$

Thank you for your attention