HOMOGENIZATION OF MULTISCALE MULTICONTINUUM SYSTEM

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Two continuum system

- $D \subset \mathbb{R}^d$ be a domain, $Y = (0,1)^d$, $\varepsilon > 0$
- Two continuum system

$$\mathcal{C}_{11}^{\varepsilon} \frac{\partial u_1^{\varepsilon}}{\partial t} = \operatorname{div}(\kappa_1^{\varepsilon} \nabla u_1^{\varepsilon}) + \frac{1}{\varepsilon^{\alpha}} Q^{\varepsilon}(u_2^{\varepsilon} - u_1^{\varepsilon}) + q,$$
$$\mathcal{C}_{22}^{\varepsilon} \frac{\partial u_2^{\varepsilon}}{\partial t} = \operatorname{div}(\kappa_2^{\varepsilon} \nabla u_2^{\varepsilon}) + \frac{1}{\varepsilon^{\alpha}} Q^{\varepsilon}(u_1^{\varepsilon} - u_2^{\varepsilon}) + q,$$

- Initial conditions: $u_1^{\varepsilon}(0) = g_1$, $u_2^{\varepsilon}(0) = g_2$.
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$$\mathcal{C}_{ii}^{\epsilon}(x) = \mathcal{C}_{ii}(x, \frac{x}{\epsilon}),\ \kappa_i^{\epsilon}(x) = \kappa_i(x, \frac{x}{\epsilon}),\ Q^{\epsilon}(x) = Q(x, \frac{x}{\epsilon}),\ \ i = 1, 2.$$

- $C_{ii}(x,y)$, $\kappa_i(x,y)$ and $Q(x,y)$ are Y-periodic wrt y.
- $\alpha = 0, 1, 2.$

Case $\alpha = 0$

- Performing two scale asymptotic expansion

$$u_1^\varepsilon(t,x) = u_{10}(t,x) + \varepsilon u_{11}(t,x,\frac{x}{\varepsilon}) + \varepsilon^2 u_{12}(t,x,\frac{x}{\varepsilon}) + \dots$$
$$u_2^\varepsilon(t,x) = u_{20}(t,x) + \varepsilon u_{21}(t,x,\frac{x}{\varepsilon}) + \varepsilon^2 u_{22}(t,x,\frac{x}{\varepsilon}) + \dots,$$

where $u_{1i}(t,x,y)$, $u_{2i}(t,x,y)$ are periodic with respect to y.

- $u_1^\varepsilon \rightharpoonup u_{10}(t,x), \quad u_2^\varepsilon \rightharpoonup u_{20}(t,x)$ in $L^2((0,T);H_0^1(D))$
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$$\langle C_{11}\rangle \frac{\partial u_{10}}{\partial t} = \mathrm{div}(\kappa_1^*\nabla u_{10}) + \langle Q\rangle(u_{20} - u_{10}) + q$$
$$\langle C_{22}\rangle \frac{\partial u_{20}}{\partial t} = \mathrm{div}(\kappa_2^*\nabla u_{20}) + \langle Q\rangle(u_{10} - u_{20}) + q$$

- κ_1^*, κ_2^* are the usual homogenized coefficients.

- $\langle\cdot\rangle$ denotes average over Y wrt y.

Case $\alpha = 1$

- Cell problems: For $k = 1, 2$, let $M_k(x, y)$ periodic in y be solution of

$$\mathrm{div}_y(\kappa_k \nabla_y M_k) + Q = 0,$$

(assuming $\int_Y Q(x, y) dy = 0$).
- For $i = 1, \dots, d$

$$\nabla_y \cdot (\kappa_k (e^i + \nabla_y N_k^i)) = 0,$$

e^i is the ith unit vector.
-

$$u_{11} = \sum_{i=1}^{d} N_1^i \frac{\partial u_{10}}{\partial x_i} + M_1(u_{20} - u_{10}),$$

$$u_{21} = \sum_{i=1}^{d} N_2^i \frac{\partial u_{20}}{\partial x_i} + M_2(u_{10} - u_{20}).$$

Case $\alpha = 1$

- Homegenized system:

$$\langle C_{11}\rangle \frac{\partial u_{10}}{\partial t} = \nabla \cdot \Big(\kappa_1^* \nabla u_{10}) + \nabla \cdot ((\int_Y \kappa_1 \nabla_y M_1 dy)(u_{20} - u_{10}) \Big) +$$
$$\sum_{i=1}^{d} \Big((\int_Y QN_2^i dy) \frac{\partial u_{20}}{\partial x_i} - (\int_Y QN_1^i dy) \frac{\partial u_{10}}{\partial x_i} \Big)$$
$$-(\int_Y Q(M_1 + M_2) dy)(u_{20} - u_{10}) + q$$

$$\langle C_{22}\rangle \frac{\partial u_{20}}{\partial t} = \nabla \cdot \Big(\kappa_2^* \nabla u_{20}) + \nabla \cdot ((\int_Y \kappa_2 \nabla_y M_2 dy)(u_{10} - u_{20}) \Big) +$$
$$\sum_{i=1}^{d} \Big((\int_Y QN_1^i dy) \frac{\partial u_{10}}{\partial x_i} - (\int_Y QN_2^i dy) \frac{\partial u_{20}}{\partial x_i} \Big)$$
$$-(\int_Y Q(M_1 + M_2) dy)(u_{10} - u_{20}) + q$$

Case $\alpha = 2$

- $u_{10} = u_{20} = u_0(t, x)$
- Cell problem

$$\nabla_y \cdot (\kappa_1(x,y)(e^i + \nabla_y N_1^i)) + Q(x,y)(N_2^i - N_1^i) = 0,$$
$$\nabla_y \cdot (\kappa_2(x,y)(e^i + \nabla_y N_2^i)) + Q(x,y)(N_1^i - N_2^i) = 0.$$

- Homogenized problem

$$(\langle C_{11}\rangle + \langle C_{22}\rangle)\frac{\partial u_0}{\partial t} = \nabla_x((\kappa_1^* + \kappa_2^*)\nabla_x u_0) + 2q.$$

- Initial condition

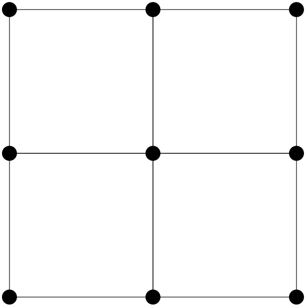
$$u_0(0,x) = \frac{\langle C_{11}\rangle g_1 + \langle C_{22}\rangle g_2}{\langle C_{11}\rangle + \langle C_{22}\rangle}.$$

Hierarchical FE for cell problems ($\alpha = 2$)

- For simplicity: the domain $D = [0,1]^d$
- Hierarchy of piecewise linear FE spaces

$$V^0 \subset V^1 \subset \ldots \subset V^L \subset H^1_{per}(Y);$$

V^l has mesh-size 2^{-l}.

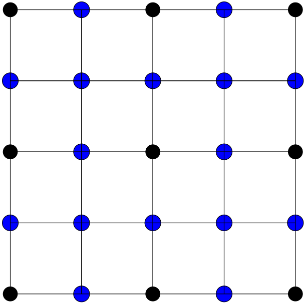


- At macroscopic points x of level 0 to solve cell problems:
  use piecewise linear FE

  with the smallest meshsize 2^{-L}, i.e. use V^L

FE error is $O(2^{-L})$.

Hierarchical FE for cell problems ($\alpha = 2$)



- At macroscopic points x of level 1: use meshsize 2^{-L+1}

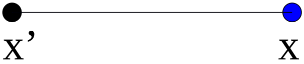
i.e use V^{L-1}

use the FE solution at points at level 0 to compensate for the larger meshsize.

Hierarchical FE for cell problems ($\alpha = 2$)

The equation to solve:

$$\nabla_y \cdot (\kappa_1(x,y)(e^i + \nabla_y N_1^i)) + Q(x,y)(N_2^i - N_1^i) = 0,$$
$$\nabla_y \cdot (\kappa_2(x,y)(e^i + \nabla_y N_2^i)) + Q(x,y)(N_1^i - N_2^i) = 0.$$



- $x' \in$ level 0, $x \in$ level 1, $\quad |x - x'| = O(1/4)$

- FE solution at x' in level 0 is $\bar{w}_i(x', \cdot)$:

$$\|N_1^i(x',\cdot) - \bar{N}_1^i(x',\cdot)\|_{H^1(Y)} \le c2^{-L}, \|N_2^i(x',\cdot) - \bar{N}_2^i(x',\cdot)\|_{H^1(Y)} \le c2^{-L}.$$

- Use $\bar{N}_1^i(x',\cdot)$ $\bar{N}_2^i(x',\cdot)$ to compute an approximation for cell problem at x, i.e. for $N_1^i(x,\cdot)$ and $N_2^i(x,\cdot)$.

Hierarchical FE for cell problems ($\alpha = 2$)

- Cell problems at x and x':

$$\nabla_y \cdot (\kappa_1(x,y)(e^i + \nabla_y N_1^i(x,y))) + Q(x,y)(N_2^i(x,y) - N_1^i(x,y)) = 0,$$
$$\nabla_y \cdot (\kappa_2(x,y)(e^i + \nabla_y N_2^i(x,y))) + Q(x,y)(N_1^i(x,y) - N_2^i(x,y)) = 0.$$

$$\nabla_y \cdot (\kappa_1(x',y)(e^i + \nabla_y N_1^i(x',y))) + Q(x',y)(N_2^i(x',y) - N_1^i(x',y)) = 0,$$
$$\nabla_y \cdot (\kappa_2(x',y)(e^i + \nabla_y N_2^i(x',y))) + Q(x',y)(N_1^i(x',y) - N_2^i(x',y)) = 0.$$

- Let

$$N_1^{ic}(x,\cdot) = N_1^i(x,\cdot) - N_1^i(x',\cdot),$$
$$N_2^{ic}(x,\cdot) = N_2^i(x,\cdot) - N_2^i(x',\cdot).$$

Hierarchical FE for cell problems ($\alpha = 2$)

- Equation for N_1^{ic} and N_2^{ic}:

$$\nabla_y(\kappa_1(x,y)\nabla_y N_1^{ic}(x,y)) + Q(x,y)(N_2^{ic}(x,y) - N_1^{ic}(x,y)) = F_1,$$
$$\nabla_y(\kappa_2(x,y)\nabla_y N_2^{ic}(x,y)) + Q(x,y)(N_1^{ic}(x,y) - N_2^{ic}(x,y)) = F_2$$

where

$$\begin{aligned} F_1 &= -\nabla_y((\kappa_1(x,y) - \kappa_1(x',y))\nabla_y N_1^i(x',y)) \\ &\quad -\nabla_y((\kappa_1(x,y) - \kappa_1(x',y))e^i) \\ &\quad +(Q(x',y) - Q(x,y))(N_2^i(x',y) - N_1^i(x',y)). \end{aligned}$$

Similar

$$\begin{aligned} F_2 &= -\nabla_y((\kappa_2(x,y) - \kappa_2(x',y))\nabla_y N_2^i(x',y)) \\ &\quad -\nabla_y((\kappa_2(x,y) - \kappa_2(x',y))e^i) \\ &\quad +(Q(x',y) - Q(x,y))(N_1^i(x',y) - N_2^i(x',y)). \end{aligned}$$

Hierarchical FE for cell problems ($\alpha = 2$)

- As FE approximation $\bar{N}_1^i(x', \cdot)$, $\bar{N}_2^i(x', \cdot)$ have been computed, we consider the same equation, with

$$
\begin{aligned}
F_1 &= -\nabla_y((\kappa_1(x,y) - \kappa_1(x',y))\nabla_y \bar{N}_1^i(x',y)) \\
&\quad -\nabla_y((\kappa_1(x,y) - \kappa_1(x',y))e^i) \\
&\quad +(Q(x',y) - Q(x,y))(\bar{N}_2^i(x',y) - \bar{N}_1^i(x',y)).
\end{aligned}
$$

Similar for F_2

- Use FE element space V^{L-1} with mesh-size 2^{-L+1},
- FE solution $\bar{N}_1^{ic}(x, \cdot)$, $\bar{N}_2^{ic}(x, \cdot)$

Hierarchical FE for cell problems ($\alpha = 0$)

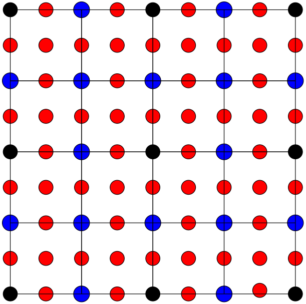
- Let

$$\bar{N}_1^i(x,\cdot) = \bar{N}_1^i(x',\cdot) + \bar{N}_1^{ic}(x,\cdot) \in V^{L-1}$$
$$\bar{N}_2^i(x,\cdot) = \bar{N}_2^i(x',\cdot) + \bar{N}_2^{ic}(x,\cdot) \in V^{L-1}.$$

- Error estimate

$$\|\bar{N}_1^i(x,\cdot) - N_1^i(x,\cdot)\|_{H^1(Y)} \le c2 \cdot 2^{-L},$$
$$\|\bar{N}_2^i(x,\cdot) - N_2^i(x,\cdot)\|_{H^1(Y)} \le c2 \cdot 2^{-L}.$$

Hierarchical macroscopic points



- At macroscopic points x of level 2: use meshsize 2^{-L+2}

i.e use V^{L-2}

use the FE solution at points at level 1 to compensate for the larger meshsize.

Hirarchical FE for cell problems ($\alpha = 2$)

- At macroscopic points x of level i:
  use V^{L-i} with meshsize 2^{-L+i}

- The FE error is $\leq ci2^{-L}$

- For all $O(2^{Ld})$ points from level 0 to L, we only use $O(L2^{dL})$ number of degrees of freedom.

- The maximum FE error is $O(L2^{-L})$

- If we solve cell problems at each point separately, then the number of degrees of freedom is $O(2^{2dL})$.

Hirarchical FE for cell problems ($\alpha = 2$)

Example: $x = (x_1, x_2) \in D = (0,1)^2$, $y = (y_1, y_2) \in Y = (0,1)^2$

$$\kappa_1(x,y) = (1 - x_1)\cos(2\pi y_1)\sin(2\pi y_2) + 3,$$
$$\kappa_2(x,y) = (1 - x_1)\sin(2\pi y_1)\cos(2\pi y_2) + 3,$$
$$Q(x,y) = (1 + x_1)\sin(2\pi y_1)\sin(2\pi y_2) + 3.$$