Generalized Multiscale Finite Element Method for Helmholtz problem in heterogeneous media

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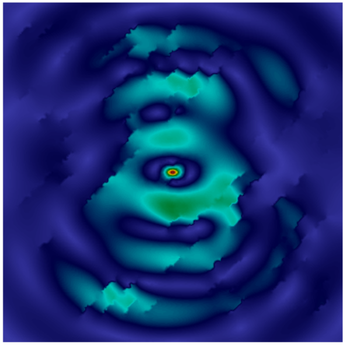
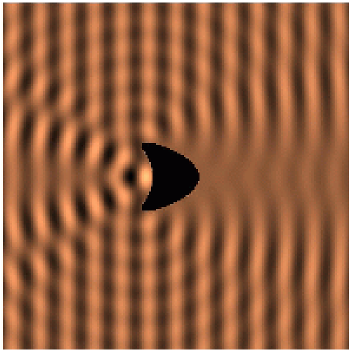
Outline

Introduction

- Complex processes in heterogeneous domains occur in many real-world applications.
- A classical numerical method for simulation of the applied problems in highly heterogeneous media should use a computational grid that resolve all small heterogeneities. For such problems, a numerical homogenization or multiscale methods are used.
- Multiscale methods should combine the simplicity and efficiency of a coarse-scale models, and the accuracy of microscale approximations.



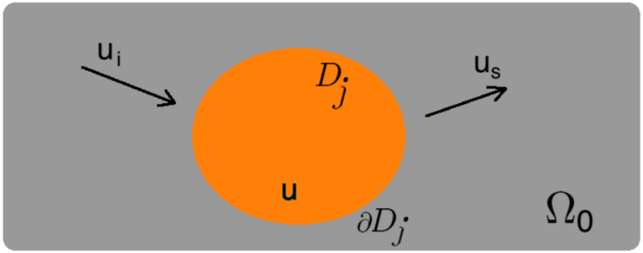


Problem formulation

Scattering problem in heterogeneous media

We consider scattering problem of a time-harmonic incident field u_i

$$\begin{aligned} \nabla(A(x)\nabla u) - k^2 n(x) u = 0, &\quad x \in D_j, \\ \Delta u_s + k^2 u_s = 0, &\quad x \in \Omega_0, \\ u_i + u_s = u, &\quad x \in \partial D_j \\ \nabla(u_i + u_s)\cdot n = A(x)\nabla u \cdot n, &\quad x \in \partial D_j \end{aligned}$$



where $u = u_i + u_s$ is the total field, u_s is the scattered field and $\Omega = \Omega_0 \cup \sum_{j=1,2,\dots} D_j$.
We consider equation with Sommerfield radiation condition

$$\frac{\partial u_s}{\partial n} - iku_s = 0, \quad x \in \partial\Omega_0$$

where n is the unit normal on boundary.

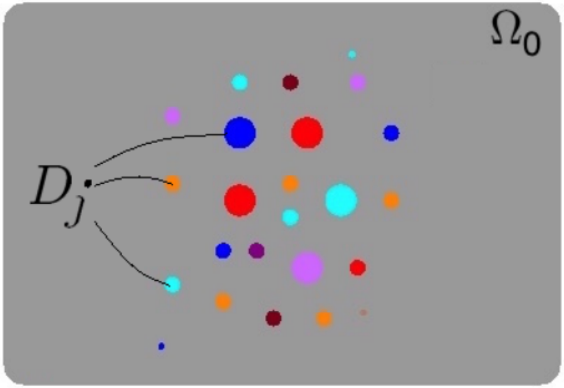
The equation above can be written as comlex-valued Helmholtz problem in domain

$$\nabla(A(x)\nabla u_s) - k^2 n(x) u_s = f(x), \quad x \in \Omega$$

with heterogeneous properties

$$n(x) = \begin{cases} 1 & x \in \Omega_0 \\ \alpha_j & x \in D_j, j = 1, 2, .. \end{cases}$$

$$A(x) = \begin{cases} \mathcal{I} & x \in \Omega_0 \\ A_j & x \in D_j, j = 1, 2, .. \end{cases}$$



and non-zero right hand side in subdomains D_j

$$f(x) = \nabla((I - A(x))\nabla u_i) + k^2(1 - n(x))u_i$$

where $u_i = e^{ikx\cdot d} = \cos(kx \cdot d) + i \sin(kx \cdot d)$ is the given incident field and d is the incident wave direction.

Fine grid approximation

Scattering problem in heterogeneous media

For approximation of the Helmholtz equation we use finite element method. We have following variational formulation: find $u \in V_h = H_1(\Omega)$ such that

$$a(u_s, v) - k^2 m(u_s, v) + ib(u_s, v) = l(v), \quad v \in V_h,$$

where

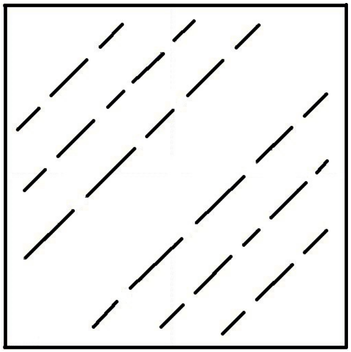
$$a(u, v) = \int_\Omega (A\nabla u, \nabla v) dx, \quad m(u, v) = \int_\Omega nuv dx,$$

$$b(u, v) = \int_{\partial\Omega} kuv ds, \quad l(v) = \int_\Omega fv dx,$$

and $u_s = Re(u_s) + iIm(u_s), \quad u_s = \sum_j U_j \phi_j, \quad \phi_j$ – linear basis functions for fine scale approximation.

Problem formulation

Helmholtz problem in fractured media



Computational domain Ω.

We consider the Helmholtz equation for the elastic waves propagation in the frequency domain

$$-\operatorname{div}\sigma(u) - \omega^2\rho u = f, \quad x \in \Omega$$

where ω is frequency, ρ is density and f is the source function.
Equation is supplemented by the relation between the stress tensor σ and strain tensor ε

$$\sigma(u) = 2\mu\varepsilon(u) + \lambda \operatorname{div} u\, E, \quad \varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$$

where E is unit tensor, λ and μ are Lame parameters.

Linear-slip model (LSM) [1]

Following the linear-slip model, we have a linear relation between traction vector and the magnitude of the discontinuity in the displacement field as follows

$$[u] = Z\sigma \cdot n,$$

where $[u]$ is the jump of the displacement field at the fracture, $\sigma \cdot n$ is the traction vector at the surface of the fracture and Z is the fracture compliance matrix.

$$Z = \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix}$$

where $z_1 = k_1^{-1}$ and $z_2 = k_2^{-1}$ are the normal and tangential compliances, respectively.

[1] Schoenberg, M. (1980). Elastic wave behavior across linear slip interfaces. The Journal of the Acoustical Society of America, 68(5), 1516-1521.

Absorbing boundary condition

In the computations, the energy of waves needs to be absorbed at artificial boundaries in order to avoid spurious reflections caused by the finite computational domain. We use a first order absorbing boundary condition

$$i\rho\omega A u = -\sigma(u) n, \quad x \in \partial\Omega,$$

where

$$A = \begin{bmatrix} n_1 & n_2 \\ -n_2 & n_1 \end{bmatrix} \begin{bmatrix} c_p & 0 \\ 0 & c_s \end{bmatrix} \begin{bmatrix} n_1 & -n_2 \\ n_2 & n_1 \end{bmatrix}.$$

Here $n = (n_1, n_2)$ is the outward normal to the boundary and c_s, c_p are the S- and P-wave velocities

$$c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_s = \sqrt{\frac{\mu}{\rho}}$$

Fine grid approximation

Helmholtz problem in fractured media

The weak formulation of the elastic wave equation for the interior penalty discontinuous Galerkin method (IPDG, Interior Penalty Discontinuous Galerkin) in fractured media is given by

$$a(u,v) - \omega^2 m(u,v) + ib(u,v) = l(v), \quad v \in V_h,$$

$$a_{DG}(u,v) = \sum_{K\in\mathcal{T}_h} \int_K (\sigma(u), \varepsilon(v))dx + \sum_{E\in\mathcal{E}_f} \int_E Z^{-1} [u] [v] \, ds -$$

$$- \sum_{E\in\mathcal{E}_c} \int_E \{\tau(u)\} [v] \, ds - \sum_{E\in\mathcal{E}_c} \int_E \{\tau(v)\} [u] \, ds + i \sum_{E\in\mathcal{E}_c} \frac{\eta}{h} \int_E (\lambda + 2\mu) [u] [v] \, ds,$$

$$m(u,v) = \sum_{K\in\mathcal{T}_h} \int_K \rho uv dx, \quad b(u,v) = \sum_{E\in\mathcal{E}_b} \int_E \rho\omega A uv ds,$$

$$l(v) = \sum_{K\in\mathcal{T}_h} \int_K f v dx.$$

where η is the penalty parameter, $\tau(u) = \sigma(u)n$ is the traction vector, $\mathcal{E}_b$ is a subset of faces on the boundary. Here $u = Re(u) + iIm(u)$, and $u = \sum_j u_j \phi_j$, ϕ_j are linear basis functions for the fine scale approximation.

Fine grid approximation for general problem

For approximation of the Helmholtz equation we use finite element method. We have following variational formulation: find $u \in V_h = H_1(\Omega)$ such that

$$a(u, v) - k^2 m(u, v) + ib(u, v) = l(v), \quad v \in V_h,$$

and $u = Re(u) + iIm(u)$.
We can write the complex valued problem in matrix form

$$(K_h + B_h - \omega^2 M_h)U = F_h,$$

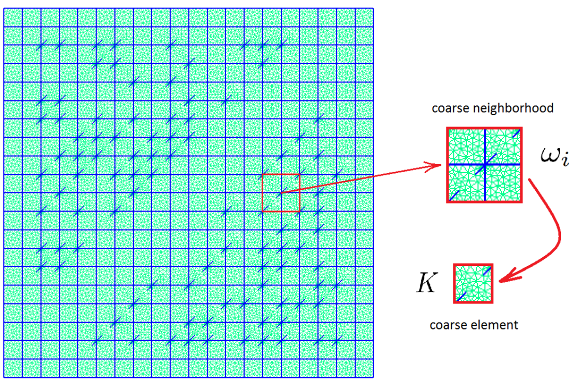
where $U = Re(U) + iIm(U)$, $(F_h)_i = l(\phi_i)$, and complex valued matrices M_h, B_h and K_h are given by

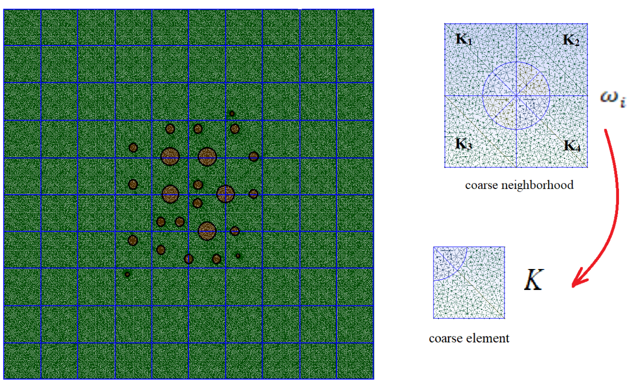
$$(M_h)_{ij} = m(\phi_i, \phi_j), \quad (B_h)_{ij} = ib(\phi_i, \phi_j), \quad (K_h)_{ij} = a(\phi_i, \phi_j).$$

Coarse grid approximation using GMsFEM

For coarse-scale approximation, we use Generalized Multiscale Finite Element Method. In the GMsFEM, we have following computational algorithm:

- the construction of the multiscale basis functions by the solution of the local eigenvalue problem in local domain ω_i and
- the construction and solution of the coarse grid approximation on multiscale space.





Coarse grid approximation using GMsFEM

The global multiscale space V_H is then defined as the linear span of all $V_H^{\omega_i}$, $\omega_i \in \mathcal{T}_H$ and will be used as the approximation space of CG approach, which can be formulated as follows: find $u_H \in V_H$:

$$a(u_H, v) - k^2 m(u_H, v) + ib(u_H, v) = l(v), \quad v \in V_H$$

For the construction of the boundary space, we solve following local spectral problem in ω_i which obtained by the combining all the coarse cells around one vertex of the coarse grid.

$$a(\phi^{\omega_i}, v) = \lambda s(\phi^{\omega_i}, v), \quad v \in V_h(\omega_i),$$

where

$$s(\phi^{\omega_i}, v) = \int_{\partial \omega_i} \rho \phi^{\omega_i} v ds$$

To construct a multiscale space $V_H^{\omega_i}$, we select the first M eigenvectors $\phi_1^{\omega_i}, \phi_2^{\omega_i}, ..., \phi_M^{\omega_i}$ corresponding to the first M smallest eigenvalues $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_M$,and define the space $V^{\omega_i, H}$ by

$$V_H^{\omega_i} = \text{span}\left\{\phi_1^{\omega_i}, \phi_2^{\omega_i}, ..., \phi_M^{\omega_i}\right\}.$$

Coarse grid approximation using GMsFEM

Matrix form. The coarse-scale system can be calculated by projecting the fine-scale matrices onto the coarse grid with the global projection matrix assembled from the calculated multiscale basis functions

$$R = (R_1, R_2, ..., R_N)^T, \quad R_j = \left[\phi_1^{\omega_i}, \phi_2^{\omega_i}, ..., \phi_M^{\omega_i}\right].$$

where R_j is the local projection matrix in a coarse element ω_j and N is the number of coarse grid elements.

$$(K_H + B_H - \omega^2 M_H)V_H = F_H,$$

where $V_H = Re(V_H) + iIm(V_H)$, M_H and K_H are the coarse-scale mass and stiffness matrices and B_H is the coarse-scale boundary mass matrix

$$M_H = RM_hR^T, \quad K_H = RK_hR^T, \quad B_H = RB_hR^T, \quad F_H = RF_h.$$

After calculation of the coarse-scale solution, we can recover the fine-scale solution

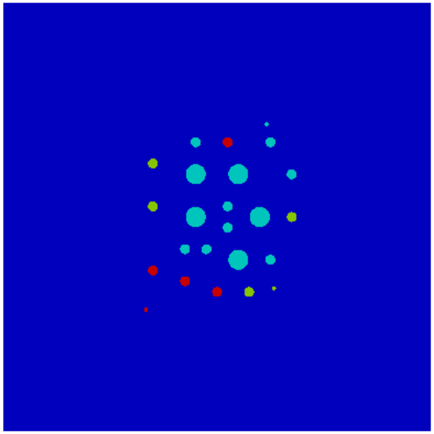
$$V_{ms} = R^T V_H.$$

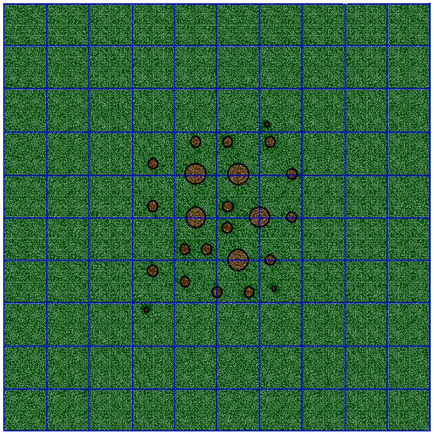
Numerical results

Scattering problem in heterogeneous media

Computational domains with heterogeneous obstacles are presented and have dimensions $\Omega = [-2, 2] \times [-2, 2]$. We set $k = 1.0$ and $d = (1, 0)$. For background, we set $\alpha = 1$ and $n = 1$. We have three types of circle inclusion:

- $\alpha = 3$, $n = 0.8$ (first),
- $\alpha = 5$, $n = 0.4$ (second),
- $\alpha = 10$, $n = 0.2$ (third)



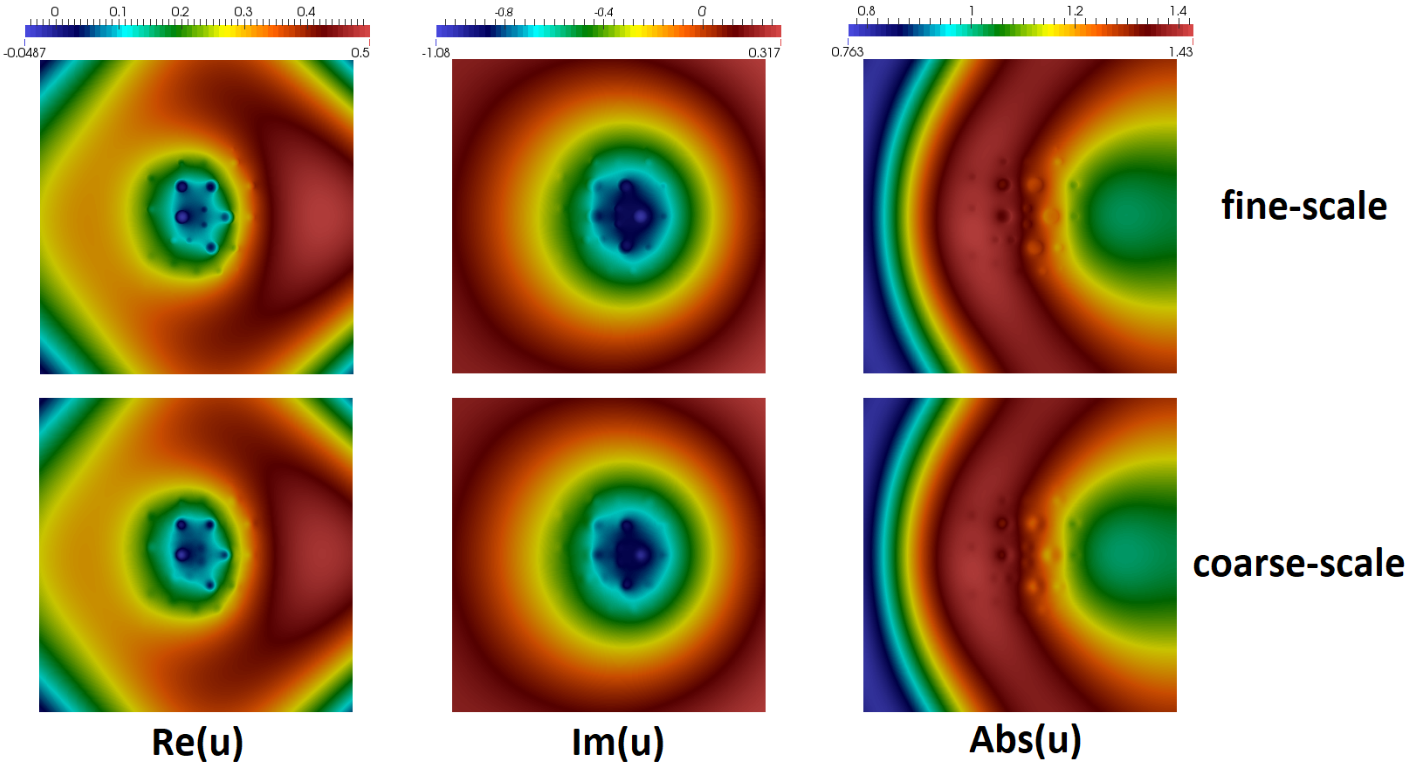


Left figure is domain with obstacles, right figure is fine mesh with 100802 vertices and 200482 triangle elements. Coarse mesh with 121 vertices and 100 triangle elements.

Numerical results

Scattering problem in heterogeneous media

Fine-scale (top) and coarse-scale (bottom) solution Re(u), Im(u), Abs(u) for heterogeneous domain with $k = 1.0$.



Numerical results

Scattering problem in heterogeneous media

Coarse mesh with 121 local domains (10×10).

M	DOF_c	$\|\|e_{u_s^{Re}}\|\|_{L_2}$ (%)	$\|\|e_{u_s^{Im}}\|\|_{L_2}$ (%)	$\|\|e_{u_s^{Abs}}\|\|_{L_2}$ (%)	time (sec)
1	121	10.9953	10.9844	3.14929	< 1
2	242	9.90949	9.9389	2.85627	< 1
4	484	7.89172	7.70215	2.28181	< 1
6	726	6.08051	6.0714	1.70784	1
8	968	5.49806	5.54987	1.5524	1
12	1452	4.45985	4.29164	1.26164	2
16	1936	2.38866	2.36736	0.663197	5

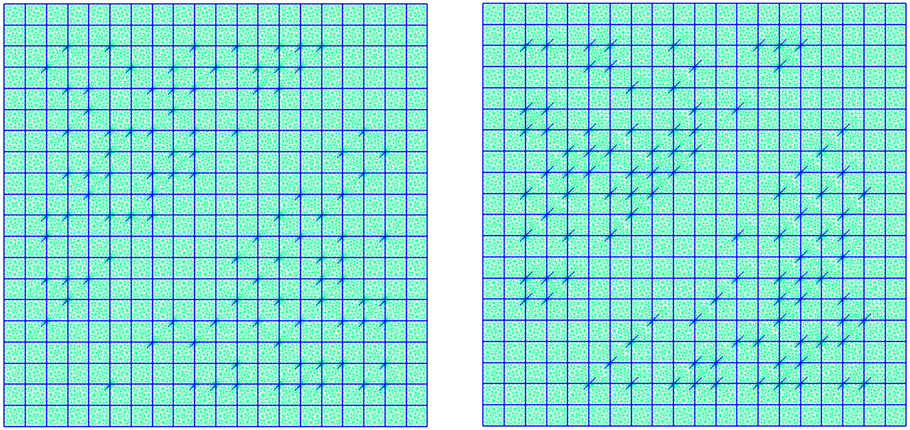
Relative errors for different number of multiscale basis. Computational time for fine scale is 28 seconds.

Numerical results

Helmholtz problem in fractured media

Computational domains with different lentgh of fracture are presented and have dimensions $\Omega = [0, L_x] \times [0, L_y]$ with $L_x = L_y = 500$[m].

- On the left of Figures, we have a fracture length of 10[m] (Case 1), and
- on the right of Figures, we have a fracture length of 20[m] (Case 2).



Computational meshes with different length (10[m], 20[m]) of fracture. Left figure is Case 1 (16077 vertices and 31752 triangle elements), right figure for Case 2 (16509 vertices and 32616 triangle elements).

We set the source term

$$f(x) = G(x)P(\theta),$$

where $P(\theta) = (cos\theta, sin\theta)$ is the polar angle of the source force vector with $\theta = 0$, $G(x) = \delta(x - x_0)$ with $x_0 = (250, 250)$ and run simulations for $\omega = 2\pi f_0$ with $f_0 = 15$ Hz.

For numerical simulation, we set following parameters

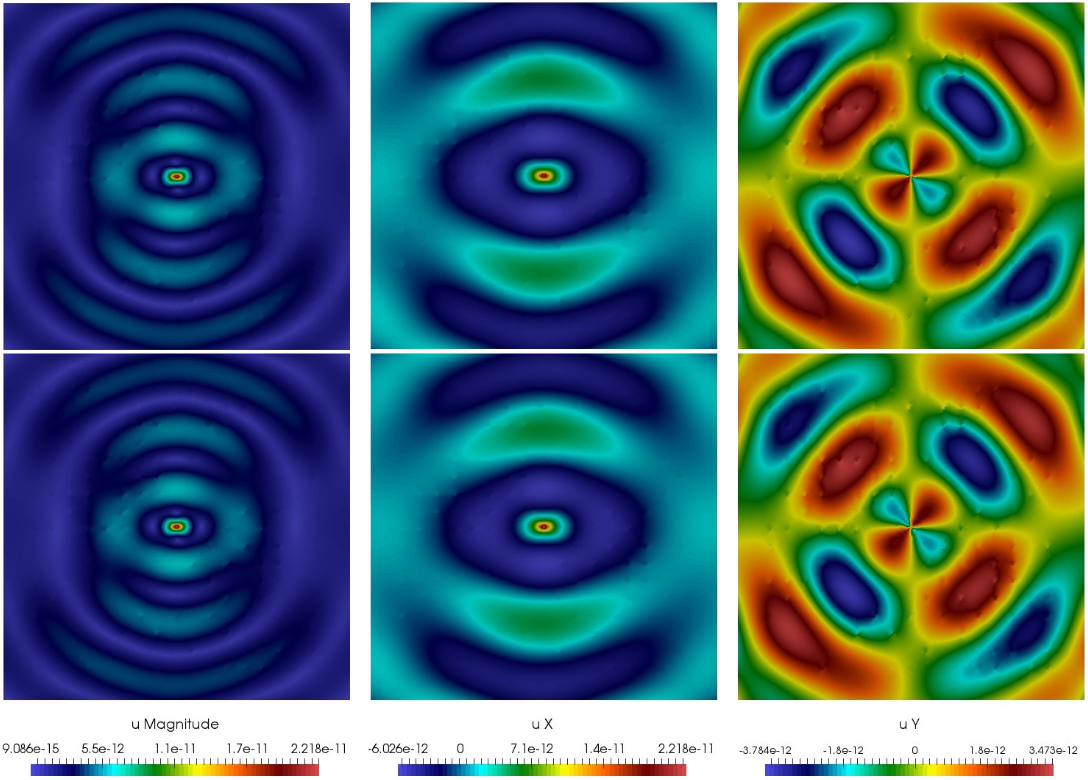
$$K = \frac{E}{3(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)},$$

with $E = 40 \cdot 10^9[\mathrm{Pa}]$, $\nu = 0.3$, $\rho = 2300[kg/m^3]$. For fracture compliance matrix Z, we use $z_1 = z_2 = 10^7[\mathrm{m/Pa}]$.

For the numerical solution, we construct structured coarse grids with 400 cells. The fine grids are unstructured grids that resolve the fractures.

Numerical results

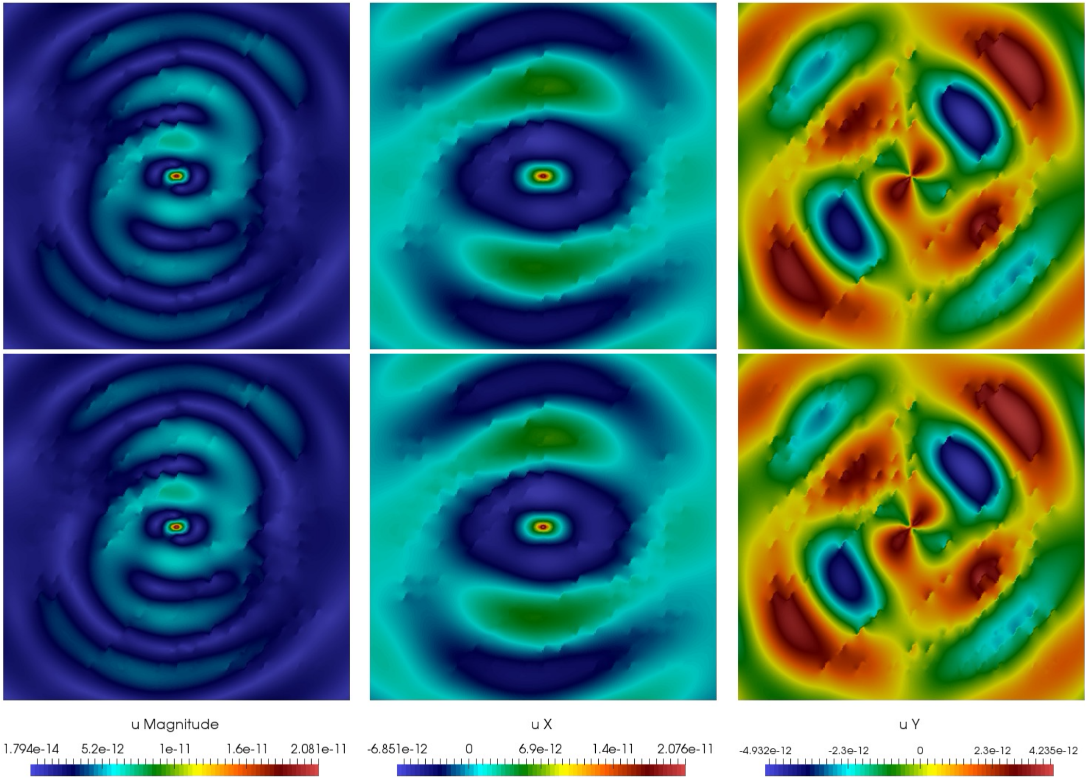
Helmholtz problem in fractured media



Fine-scale (top) and coarse-scale (bottom) solution for Magnitude, X, Y (from left to right) for domain with fracture (Case 1).

Numerical results

Helmholtz problem in fractured media



Fine-scale (top) and coarse-scale (bottom) solution for Magnitude, X, Y (from left to right) for domain with fracture (Case 2).

Numerical results

Helmholtz problem in fractured media

M	DOF_c	$f_0 = 5$ (Hz)	$f_0 = 10$ (Hz)	$f_0 = 15$ (Hz)	time(sec)
4	484	26.1354	35.2117	70.3659	< 1
6	726	22.7009	29.2749	53.8333	< 1
8	968	14.6834	18.319	30.5928	1
10	1210	11.2943	14.2606	23.8355	3
12	1452	8.77733	10.9336	17.595	6
16	1936	4.45949	7.62178	10.6931	13
20	2420	4.1722	5.67664	9.17952	21

Relative errors L_2 norm (%) for displacement for different number of multiscale basis. $\omega = 2\pi f_0$. $DOF_f = 195696$. Computational time for fine scale is about 270 seconds.

Conclusion

- We considered a Helmholtz equation in fractured medium using linear-slip model (LSM) and scattering problem in heterogeneous medium.
- We constructed the reduced order model using Generalized Multiscale Finite Element Method.
- We present numerical results for geometries with fractures and obstacles.
- Our results show that the presented method give good approximation of the solution and reduce size of system.
- In future we will consider another types of multiscale basis construction and 3D problems.

Thank you for your attention!