Homogenization of the Steklov Spectral Problem for the System of Elasticity

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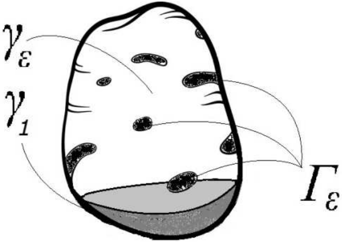
The Domain

Let Ω be a smooth domain in $\mathbb{R}^d$, $d \geqslant 2$.



We suppose that the boundary $\partial\Omega = \gamma_1 \cup \gamma_\varepsilon \cup \Gamma_\varepsilon$. Γ_ε consists of the sets Γ_ε^m, $m = 1, \ldots, N_\varepsilon$, the diameter of Γ_ε^m is less then or equals to ε, and the distance between them is greater than or equals to 2ε, where ε is a positive small parameter.

Problem setup

Consider the spectral problem

$$\begin{cases} L_k(u_\varepsilon^n) := \frac{\partial}{\partial x_i}\left(a_{kl}^{ij}\frac{\partial u_\varepsilon^l}{\partial x_j}\right) = 0 \text{ in } \Omega, \; k = 1, \dots, d, \\ u_\varepsilon^n = 0 \text{ on } \gamma_1 \cup \gamma_\varepsilon, \\ \sigma(u_\varepsilon^n) := A^{ij}(x)\frac{\partial u_\varepsilon}{\partial x_j}\nu_i = \lambda_\varepsilon^n u_\varepsilon^n \text{ on } \Gamma_\varepsilon, \; n = 1, 2, \dots \end{cases} \tag{1}$$

Here and throughout we assume the summation on the repeated indices.
Here $u_\varepsilon^n \in (H^1(\Omega, \gamma_1 \cup \gamma_\varepsilon))^d$, $n = 1, 2, \dots$
The set $\{\lambda_\varepsilon^n\}$, $n = 1, 2, \dots,$ is the set of eigenvalues such that $\lambda_\varepsilon^1 \leqslant \lambda_\varepsilon^2 \leqslant \cdots \leqslant \lambda_\varepsilon^n \leqslant \cdots$, where the eigenvalues repeat according to their multiplicities.

Problem setup

A^{ij} are matrices $(d \times d)$ with elements a_{kl}^{ij}, which are bounded measurable functions, $a_{kl}^{ij}(x) = a_{lk}^{ji}(x) = a_{il}^{kj}(x)$,

$$\varkappa_1 \xi_{ki}\xi_{ki} \leqslant a_{kl}^{ij}(x)\xi_{ki}\xi_{lj} \leqslant \varkappa_2 \xi_{ki}\xi_{ki}, \quad \varkappa_1, \varkappa_2 = const > 0, \quad x \in \Omega,$$

where $\{\xi_{ki}\}$ are real symmetric matrices, $\nu = (\nu_1, \dots, \nu_d)$ is an outward normal vector to the boundary $\partial\Omega$.

Problem setup

We study the limit behavior of eigenelements of problem (1), when ε tends to zero and $N_\varepsilon = O(|\ln\varepsilon|^{(1-\frac{\delta}{2})d-1}), 0 < \delta < 2 - \frac{2}{d}$, where N_ε is the number of Γ_ε^m on the boundary. The space $H^1(\Omega, \gamma_1 \cup \gamma_\varepsilon)$ is defined as the completion of the functions from the space $C^\infty(\overline{\Omega})$, vanishing in a neighborhood of $\gamma_1 \cup \gamma_\varepsilon$, with respect to the norm

$$\|v\|_{H^1(\Omega)} = \Big(\int\limits_\Omega \big(v^2 + |\nabla v|^2\big)\,dx\Big)^{\frac{1}{2}}.$$

Auxiliary Propositions

Consider the boundary-value problem corresponding to spectral problem (1). We have

$$\begin{cases} L(u_\varepsilon) = 0 \quad \text{in} \quad \Omega, \\ u_\varepsilon = 0 \quad \text{on} \quad \gamma_1 \cup \gamma_\varepsilon, \\ \sigma(u_\varepsilon) = g(x) \quad \text{on} \quad \Gamma_\varepsilon, \end{cases} \tag{2}$$

where $g(x) \in \Big(L_2(\partial\Omega)\Big)^d$, $u_\varepsilon = (u_{\varepsilon,1}, \ldots, u_{\varepsilon,d})^T$ and

$$L(u) = (L_1(u), \ldots, L_d(u))^T := \frac{\partial}{\partial x_i}\big(A^{ij}(x)\frac{\partial u_\varepsilon}{\partial x_j}\big).$$

Auxiliary Propositions

Let $\Omega_\eta = \{x : x \in \Omega, \rho(x, \partial\Omega) \leqslant \eta\}$, where $\rho(x, \partial\Omega)$ is the distance between x and $\partial\Omega$.

Lemma 1

For the functions u from the space $H^1(\Omega, \gamma_1 \cup \gamma_\varepsilon)$ the following estimate

$$\int\limits_{\Omega_\eta} u^2 dx \leqslant C_1 \eta^2 \int\limits_{\Omega_\eta} |\nabla u|^2 dx$$

holds true, where the constant C_1 does not depend on ε, η and u.

Auxiliary Propositions

Lemma 2

For the function u from the space $H^1(\Omega, \gamma_1 \cup \gamma_\varepsilon)$ the estimate

$$\int\limits_{\Omega} u^2 dx \leqslant C_2 \int\limits_{\Omega} |\nabla u|^2 dx,$$

is valid, where the constant C_2 does not depend on ε and u.

Auxiliary Propositions

The next Theorem is connected with Korn's inequality for strains and stresses.

Lemma 3 (Korn's type inequality)

For the function $u(x) = (u^1(x), \ldots, u^d(x))^T$ from the space $\left(H^1(\Omega, \gamma_1 \cup \gamma_\varepsilon)\right)^d$ the inequality

$$\int\limits_\Omega \sum_{i=1}^{d} |\nabla u^i|^2 dx \leqslant C_3 \int\limits_\Omega \sum_{i,j=1}^{d} \Big(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\Big)^2 dx \leqslant$$

$$\leqslant C_4 \int\limits_\Omega \sum_{i,j,k,l=1}^{d} a_{kl}^{ij}(x) \frac{\partial u^l}{\partial x_j} \frac{\partial u^k}{\partial x_i} dx$$

holds true, where C_3, C_4 are independent of u and ε.

Auxiliary Propositions

Lemma 4

If the domain G is star–shaped with respect to the ball Q, then the following Korn's type inequality

$$D(u, G) \leqslant C_5 \big(E(u, G) + D(u, Q) \big)$$

is valid, where C_5 is a constant, which does not depend on u, E and D are the following functions:

$$D(u, \Omega) \equiv \int\limits_{\Omega} \sum_{i=1}^{d} |\nabla u^i|^2 dx, \quad E(u, \Omega) \equiv \int\limits_{\Omega} \sum_{i,j=1}^{d} \Big(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\Big)^2 dx.$$

Auxiliary Propositions

Lemma 5

The solutions u_ε of the problem (2) are uniformly bounded with respect to ε in $H^1(\Omega)$.

Now define the cut-off function $\psi(s) \in C^\infty(\mathbb{R})$ such that $\psi(s) = 0$, when $s \in [-\infty, 1]$, $\psi(s) = 1$, when $s \geqslant 1 + \sigma, 0 < \sigma < \frac{1}{2}, 0 \leqslant \psi(s) \leqslant 1$.

Auxiliary Propositions

Lemma 6

For the solutions u_ε of the problem (2) the estimate

$$\int\limits_{\Omega} |\nabla u_\varepsilon|^2 \psi_\varepsilon^2(x)dx \leqslant C_6 |\ln\,\varepsilon|^{-\delta}$$

is valid, where the constant C_6 does not depend on ε;
$\psi_\varepsilon(x) = \prod\limits_{m=1}^{N_\varepsilon} \psi_\varepsilon^m(x), \psi_\varepsilon^m(x) = \psi\left(\frac{|\ln \varepsilon|}{|\ln r_m|}\right)$, where $(r_m, \theta_m^1, \dots, \theta_m^{d-1})$
is a local system of polar coordinates with the center in $p_\varepsilon^m \subset \Gamma_\varepsilon^m$.

Auxiliary Propositions

Lemma 7

For the solution u_ε of the problem (2) the estimate

$$\int\limits_\Omega |u_\varepsilon|^2 dx \leqslant C_7 |\ln \varepsilon|^{-\delta},\ 0 < \delta < 2 - \frac{2}{d},$$

is valid.

Now we use the Oleinik –Iosifian–Shamaev method to estimate the eigenvalues. Let $\boldsymbol{H}_\varepsilon$ and $\boldsymbol{H}_0$ be two separable Hilbert spaces with the scalar products $(\cdot,\cdot)_\varepsilon$ and $(\cdot,\cdot)_0$, respectively. Let $\boldsymbol{A}_\varepsilon \in \mathcal{L}(\boldsymbol{H}_\varepsilon)$ and $\boldsymbol{A}_0 \in \mathcal{L}(\boldsymbol{H}_0)$ be linear operators. Let $\boldsymbol{V}$ be a linear subspace of $\boldsymbol{H}_0$ such that $\{v \,:\, v = \boldsymbol{A}_0 u, \;\; u \in \boldsymbol{H}_0\} \subset \boldsymbol{V}$.

Let $\{\mu_\varepsilon^n\}_{n=1}^\infty$ and $\{\mu_0^n\}_{n=1}^\infty$ be the sequences of the eigenvalues of $\boldsymbol{A}_\varepsilon$ and $\boldsymbol{A}_0$, respectively, with the classical convention of repeated eigenvalues. Let $\{w_\varepsilon^n\}_{n=1}^\infty$ and $(\{w_0^n\}_{n=1}^\infty$, respectively) be the corresponding eigenfunctions in $\boldsymbol{H}_\varepsilon$, which are assumed to be orthonormal ($\boldsymbol{H}_0$, respectively).

Auxiliary Propositions
General Method

We assume that the following properties are satisfied:

- $C1$ There exists $\boldsymbol{R}_\varepsilon \in \mathcal{L}(\boldsymbol{H_0}, \boldsymbol{H}_\varepsilon)$ such that $(\boldsymbol{R}_\varepsilon F, \boldsymbol{R}_\varepsilon F)_{\boldsymbol{H}_\varepsilon} \xrightarrow{\varepsilon \to 0} \varkappa_0 (F, F)_{\boldsymbol{H_0}}$, for all $F \in \boldsymbol{V}$ and certain positive constant $\varkappa_0$.
- $C2$ The operators $\boldsymbol{A}_\varepsilon$ and $\boldsymbol{A_0}$ are positive, compact and selfadjoint. Moreover, $\|\boldsymbol{A}_\varepsilon\|_{\mathcal{L}(\boldsymbol{H}_\varepsilon)}$ are bounded by a constant, independent of ε.
- $C3$ $\|\boldsymbol{A}_\varepsilon \boldsymbol{R}_\varepsilon F - \boldsymbol{R}_\varepsilon \boldsymbol{A_0} F\|_{\boldsymbol{H}_\varepsilon} \xrightarrow{\varepsilon \to 0} 0$ for all $F \in \boldsymbol{V}$.
- $C4$ The family of operators $\boldsymbol{A}_\varepsilon$ is uniformly compact, i.e., for any sequence F^ε in $\boldsymbol{H}_\varepsilon$ such that $\sup_\varepsilon \|F^\varepsilon\|_{\boldsymbol{H}_\varepsilon}$ is bounded by a constant independent of ε, we can extract a subsequence $F^{\varepsilon'}$, that verifies the following: $\|\boldsymbol{A}_{\varepsilon'} F^{\varepsilon'} - \boldsymbol{R}_{\varepsilon'} v^0\|_{\boldsymbol{H}_{\varepsilon'}} \to 0$, as $\varepsilon' \to 0$, for certain $v^0 \in \boldsymbol{H_0}$.

Then, for each k, there exists a constant C_8^k, independent of ε, such that

$$|\mu_\varepsilon^k - \mu_0^k| \leqslant C_8^k \sup_{\substack{u \in \mathcal{N}(\mu_0^k, \boldsymbol{A_0}), \\ \|u\|_{\boldsymbol{H_0}} = 1}} \|\boldsymbol{A}_\varepsilon \boldsymbol{R}_\varepsilon u - \boldsymbol{R}_\varepsilon \boldsymbol{A_0} u\|_{\boldsymbol{H}_\varepsilon}, \tag{3}$$

where $\mathcal{N}(\mu_0^k, \boldsymbol{A_0}) = \{u \in \boldsymbol{H_0}, \boldsymbol{A_0} u = \mu_0^k u\}$. Moreover, if μ_0^k has multiplicity s ($\mu_0^k = \mu_0^{k+1} = \cdots = \mu_0^{k+s-1}$), then for any w — eigenfunction associated with μ_0^k, with $\|w\|_{\boldsymbol{H_0}} = 1$ — there exists a linear combination w^ε of eigenfunctions of $\boldsymbol{A}_\varepsilon$, $\{w_\varepsilon^j\}_{j=k}^{j=k+s-1}$ associated with $\{\mu_\varepsilon^j\}_{j=k}^{j=k+s-1}$ such that

$$\|w^\varepsilon - \boldsymbol{R}_\varepsilon w\|_{\boldsymbol{H}_\varepsilon} \leqslant C_9^k \|\boldsymbol{A}_\varepsilon \boldsymbol{R}_\varepsilon w - \boldsymbol{R}_\varepsilon \boldsymbol{A_0} w\|_{\boldsymbol{H}_\varepsilon},$$

where the constant C_9^k is independent on ε.

Main Result

Define the operator $\boldsymbol{A}_\varepsilon : L_2(\partial\Omega)^d \to H^1(\Omega, \gamma_1 \cup \gamma_\varepsilon)^d$, setting $\boldsymbol{A}_\varepsilon g = u_\varepsilon\big|_{\partial\Omega}$, where u_ε is the solution of the problem (2). The operator $\boldsymbol{A}_0$ is the zero operator. Let $\boldsymbol{H}_\varepsilon = \boldsymbol{H}_0 = L_2(\partial\Omega), \boldsymbol{V} = H^1(\Omega, \partial\Omega)$ and let $\boldsymbol{R}_\varepsilon$ be an identical operator in $L_2(\partial\Omega)$.

Let us verify the conditions C1–C4. The condition C1 is fulfilled automatically. It is easy to establish the positiveness, self–adjointness and compactness of the operators $\boldsymbol{A}_\varepsilon$. The norms $\|\boldsymbol{A}_\varepsilon\|_{\mathcal{L}(\boldsymbol{H}_\varepsilon)}$ are uniformly bounded with respect to ε by virtue of Lemma 5. Here $\mathcal{L}(\boldsymbol{H}_\varepsilon)$ is the space of linear operators on $\boldsymbol{H}_\varepsilon$.

In view of Lemma 7 the condition C3 takes place. If a sequence $\{\boldsymbol{A}_\varepsilon f_\varepsilon\}$ is bounded in $H^1(\Omega, \gamma_1 \cup \gamma_\varepsilon)$, therefore, it is compact in $L_2(\Omega)$. Because of Lemma 5 the condition C4 is fulfilled.

Main Result

Consider the spectral problem

$$\boldsymbol{A}_{\boldsymbol{\varepsilon}} u_\varepsilon^n = \mu_\varepsilon^n u_\varepsilon^n,\ \mu_\varepsilon^1 \geqslant \mu_\varepsilon^2 \geqslant \cdots,\ n = 1, 2, \ldots$$

It is obvious, that $\mu_\varepsilon^n = \dfrac{1}{\lambda_\varepsilon^n}$, where λ_ε^n is the n-th eigenvalue of the problem (1). Then formula (3) of Oleinik–Iosifian–Shamaev method gives us:

$$|\mu_\varepsilon^n| \leqslant C_{10} \sup_{\|u\|_{H_0=1}} \|\boldsymbol{A}_{\boldsymbol{\varepsilon}} u\|_{\boldsymbol{H}_{\boldsymbol{\varepsilon}}}, \tag{4}$$

$n = 1, 2, \ldots$.

Thus the next Theorem follows from (4) and Lemma 7:

Main Result

Theorem

There exists such a constant C independent of ε, that for eigenvalues λ_ε^n of the problem (1), the estimate

$$\lambda_\varepsilon^n \geqslant C|\ln \varepsilon|^\delta$$

is valid for sufficiently small ε, where
$0 < \delta < 2 - \frac{2}{d}$, $N_\varepsilon = O\left(|\ln \varepsilon|^{\left(1-\frac{\delta}{2}\right)d-1}\right)$ as $\varepsilon \to 0$.

The Bibliography

- A. Chechkina, C. D'Apice, U. De Maio. Rate of Convergence of Eigenvalues to Singularly Perturbed Steklov-Type Problem for Elasticity System. *Applicable Analysis* DOI: 10.1080/00036811.2017.1416104

Thank you for your attention!