Numerical simulation of the transport and flow problems in perforated domains using Multiscale model reduction

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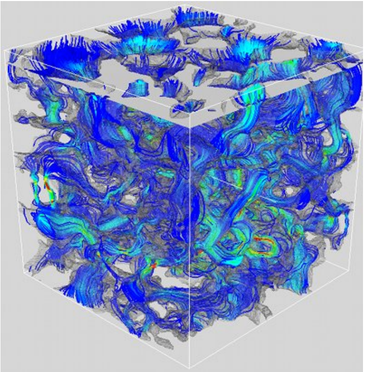
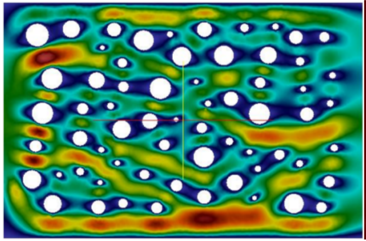
Moscow. 2018

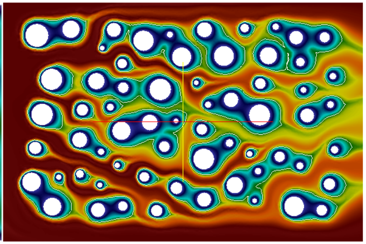
Outline

Introduction

- Many processes in real applications have multiscale nature.
- In these physical processes, the transport of the material can be described by the convection-diffusion equation.
- Numerical solutions for flow and transport equations are expensive and require resolving fine-scale details.
- We design of a general multiscale finite element framework that takes advantage of the effective low dimensional solution space for multiscale problems.

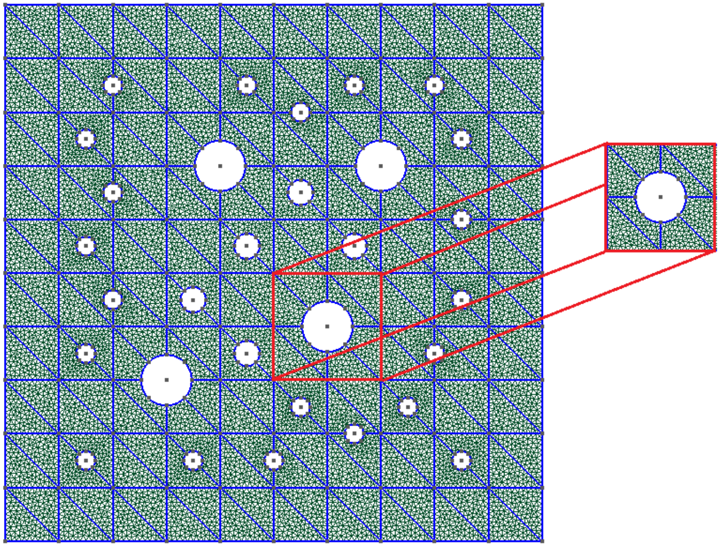






Introduction

- Among multiscale problems, the problems in perforated domains are of great interest to many applications.
- The main characteristics of these problems is that the underlying processes occur in multiscale domains where the geometry of the domain has multiple scales.
- The solution techniques require high resolution. In particular, the discretization needs to honor the irregular boundaries of perforations. This gives rise to a fine-scale problems with many degrees of freedom which can be very expensive to solve.



Mathematical model

Convection-diffusion equation

$$\frac{\partial c}{dt} - \operatorname{div}(d\nabla c) + u\nabla c = f, \quad x \in \Omega, t > 0$$

where d - diffusive transfer coefficient, u - the velocity of fluid flow in a porous medium, f - source term.

Stokes equations

$$\mu\Delta u - \nabla p = 0, \quad x \in \Omega^{\varepsilon}$$
$$\operatorname{div} u = 0, \quad x \in \Omega^{\varepsilon}$$

where μ - viscosity, p - pressure.

Mathematical model

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where d - diffusive transfer coefficient, u - the velocity of fluid flow in a porous medium, f - source term.
We will use mixed formulation for the transport equation and write formulation associated to flux. Let $q = -d\nabla c$ be the flux then we have following mixed formulation for flux and concentration (q, c)

$$d^{-1}q + \nabla c = 0, \quad x \in \Omega^\varepsilon$$
$$\frac{\partial c}{\partial t} + \operatorname{div} q - d^{-1}uq = f, \quad x \in \Omega^\varepsilon$$

Fine-grid approximation

Space discretization of the mixed transport equations

The variational formulation of the transport equation in the mixed formulation reads: find $q \in V_0^q$ and $c \in Q^c$ such that

$$\int_\Omega d^{-1}(q, z)dx + \int_\Omega \operatorname{div} z c^{n+1} dx = 0$$

$$\int_\Omega \frac{c^{n+1} - c^n}{\tau} r dx + \int_\Omega \operatorname{div} q r dx - \int_\Omega d^{-1}(ur, q)dx = \int_\Omega f r dx$$

Space discretization of the Stokes equations

$$\mu \int_\Omega \nabla u \nabla v - \int_\Omega p \operatorname{div} v dx = \int_\Omega f v dx$$

$$\int_\Omega \operatorname{div} u q dx = 0$$

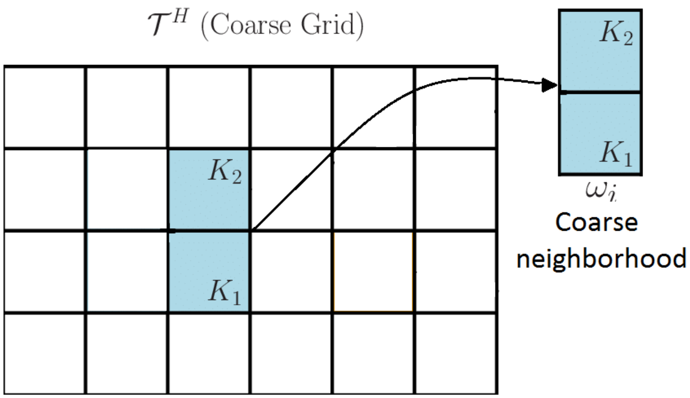
Coarse grid approximation

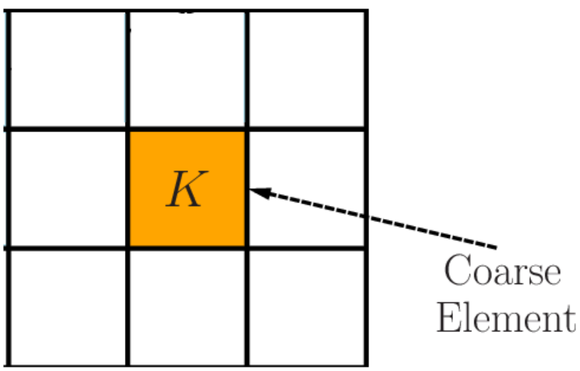
Offline computations:

- Generate a coarse grid $\mathcal{T}_H$
- Construct of snapshot space
- For each coarse region compute local snapshots and reduce the dimension of local snapshot space using a spectral decomposition

Online computations

- Construct multiscale basis functions by the solution of the local eigenvalue problem in each local domain ω_i.
- Construct and solve of the coarse grid approximation





Coarse grid approximation

Let ε_H be the set of all edges of the coarse grid and N_E be the total number of edges, ω_i is the local domain, where $i = 1, ..., N_f$ and N_f is the number of coarse grid nodes. A local domain ω_i is obtained by the combining all the coarse cells around one facets of the coarse grid. We solve following problem on the local domain w that correspond to the coarse-grid edge $E \in \varepsilon^H$: find $(\phi_j \eta) \in V_h^\omega \times Q_h^\omega$ such that

$$\int_\Omega d^{-1} \phi_j v dx - \int_\Omega \eta \operatorname{div} v dx = 0, v \in {V^\omega}_h$$

$$\int_\Omega r \nabla \cdot \phi_j dx = \int_\Omega g r dx, r \in Q_h^\omega$$

with boundary condition

$$\phi_j \cdot n = 0 \; on \; \partial\omega$$

The constant g is chosen by compatibility condition, that is, $g = \frac{1}{|K_j|} \int_E \phi_j \cdot n ds$

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with boundary condition

$$\phi_j \cdot n = 0 \quad on\ \partial\omega$$

On the coarse edge we set the additional boundary condition

$$\phi_j \cdot n = \delta_j \quad on\ E$$

where $j = 1, ..., L_\omega$. L_ω is the number of fine grid edges. Here δ_j is a piecewise constant function defined on E that has value 1 on e_j and value 0 on the order fine-grid edges. The constant g is chosen by compatibility condition, that is, $g = \frac{1}{|K_j|} \int_E \phi_j \cdot n ds$

Coarse grid approximation

The collection of the solutions of above local problems generated the snapshot space in ω

$$R_\omega = [\phi_1, ...\phi_{L_\omega}]$$

We consider following local spectral problem in the snapshot space

$$\overline{A}_\omega \overline{\psi}^\omega{}_k = \lambda_k \overline{S}_\omega \overline{\psi}_k^\omega$$

where $\overline{A}_\omega = R_\omega A_\omega R_\omega^T, \overline{S} = R_\omega S_\omega R_\omega^T$ and

$$A_\omega = [a_{mn}^\omega], a_{mn}^\omega = a_\omega(\phi_m, \phi_n) = \int_E k^{-1}(\phi_m \cdot n)(\phi_n \cdot n) ds,$$

$$S_\omega = [m_{mn}^\omega], s_{mn}^\omega = s_\omega(\phi_m, \phi_n) = \int_\omega k^{-1} \phi_m \phi_n dx + \int_\omega \nabla \cdot \phi_m \nabla \cdot \phi_n dx.$$

Coarse grid approximation

To construct a multiscale space V^{ω_i} we select the first L_i eigenvectors $\phi_1, \phi_2, ..., \phi_\omega$, corresponding to the first M_ω smallest eigenvalues $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_\omega$. We can solve eigenvalue problem for the matrix $\overline{S}^\omega$ and choose the largest eigenvalues and take corresponding eigenvectors as multiscale basis functions. The coarse-scale system can be calculated by projecting the fine-scale matrices onto coarse grid with global projection matrix assembled from the calculated multiscale basis functions

$$R = (R_1, R_2, ..., R_{N_v})^T, R_i = [\phi_1^i, \phi_2^i, ..., \phi_\omega^i],$$

where R_i is the local projection matrix. Using the global projection matrix R, we can define the coarse-scale system

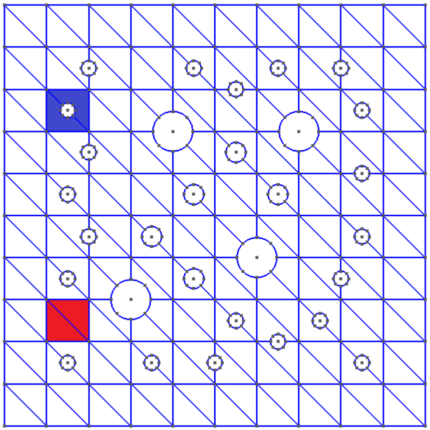
$$A_c u_c = F_c, \quad A_c = RAR^T, \quad F_c = RF.$$

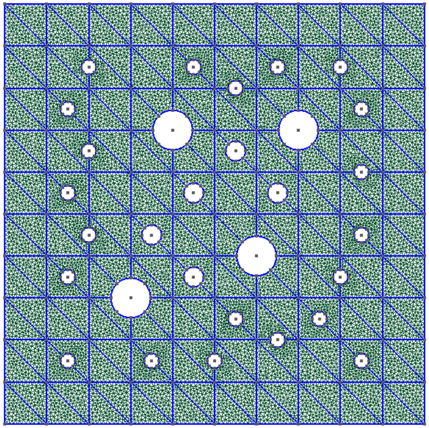
After calculation of the coarse-scale solution u_c, we can recover the multiscale solution.

$$u_{ms} = R^T u_c$$

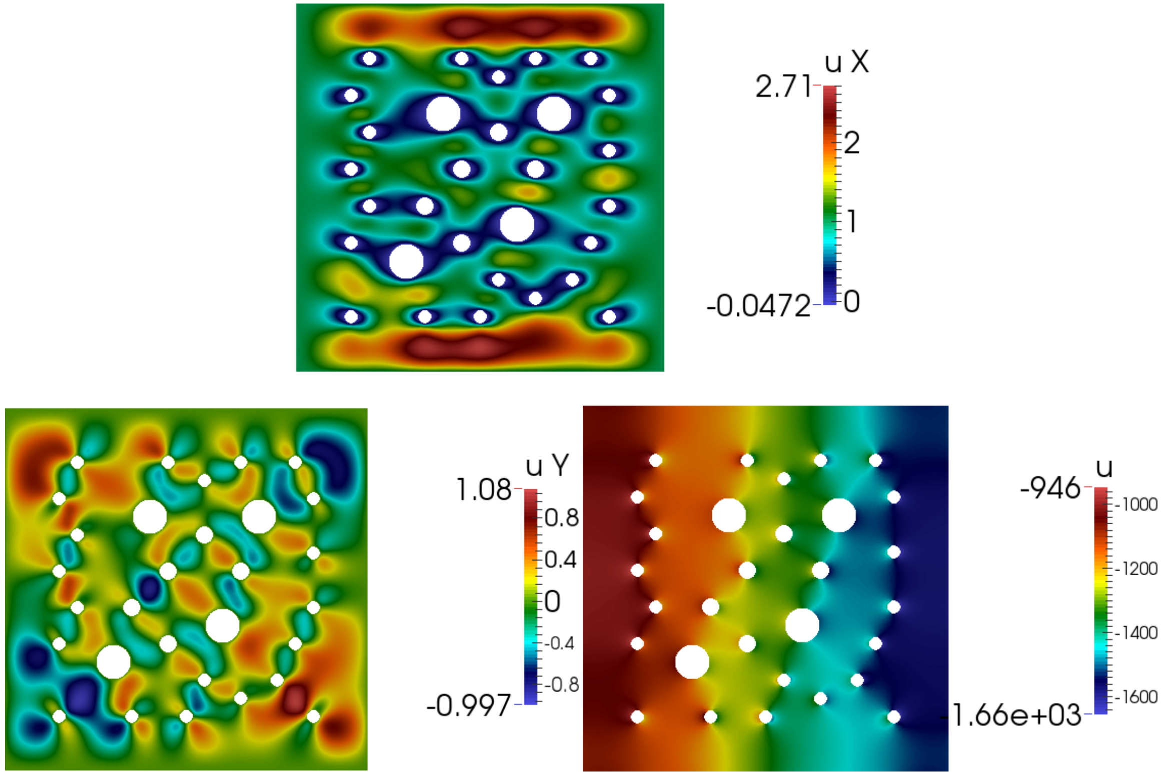
Numerical Results

- The computational mesh contains 14648 vertices, 43087 facets and 28410 cells.
- Parameters: $\mu = 1$, $dt = 0.1$, $T_{max} = 1.5$, $d = 0.03$ and source terms (see left picture in Figure 1, where blue block corresponds to $f = -1$ and orange block corresponds to $f = 1$, and $f = 0$ elsewhere in the domain).
- Initial condition c = 0



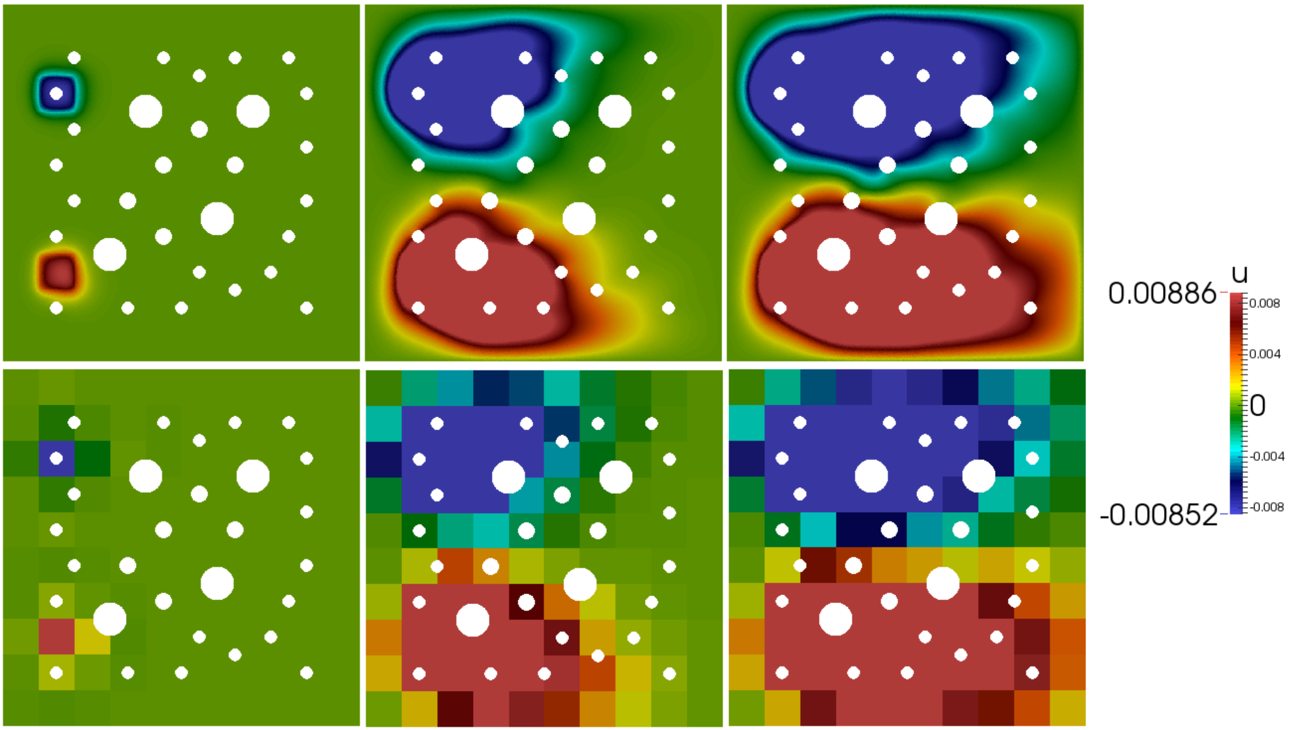


Numerical results



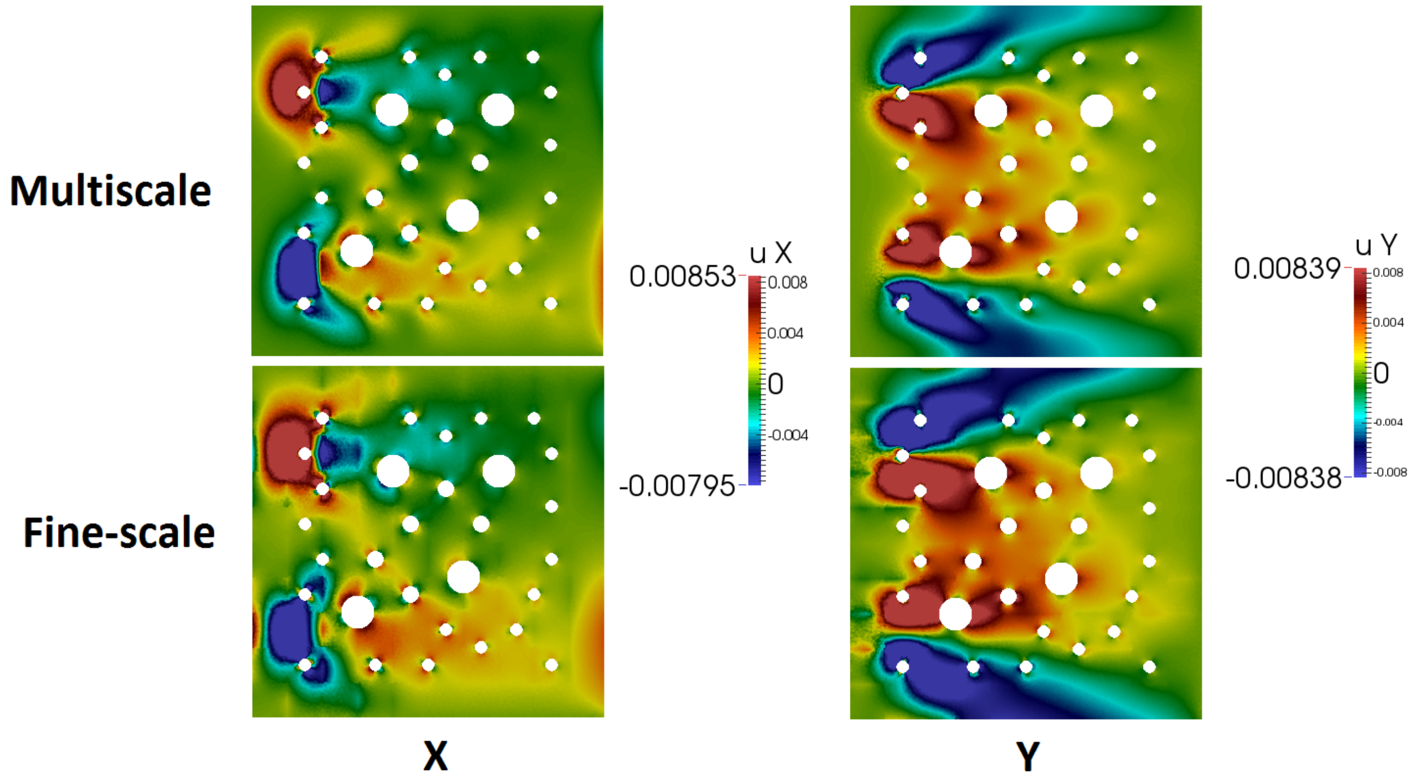
The distribution of the X and Y components of the velocity and pressure(u_x - top, u_y - left, p - right)

Numerical results



The distribution of the concentration at t=0.1sec., t=0.75 sec., t=1.5 sec. respectively(top:fine-scale solution and bottom:multiscale solution for d=0.03).

Numerical results



The distribution of the flux at t=1.5 sec. for d=0.03.

Numerical results

M	DOF_c	$e^q_{L_2}$ (%)	$e^q_{H_1}$ (%)	$e^c_{L_2}$ (%)	$e^{\bar{c}}_{L_2}$ (%)
1	320	10.024	27.7091	6.65845	0.868823
2	540	7.33061	27.0764	6.65333	0.868823
3	760	6.03289	27.0647	6.65083	0.701797
4	980	5.99444	27.0618	6.65082	0.726132
1	320	5.15484	0.844318	7.95336	0.031527
2	540	0.60391	0.7261	7.9223	0.006332
3	760	0.22771	0.72225	7.92101	0.005091
4	980	0.19374	0.721922	7.92099	0.005104
1	320	5.44966	0.0148155	8.71128	0.194684
2	540	0.440545	0.00775466	8.60703	0.0013234
3	760	0.0731469	0.00739732	8.60553	0.0013234
4	980	0.0380817	0.00736612	8.60552	0.000262544

Relative errors for flux and concentration with different number of multiscale basis functions for $d = 0.03$, $d = 0.3$ and $d = 3.0$ sequentially for $t = 1.5$

Conclusion

- We considered a multiscale model reduction approach based on mixed GMsFEM for the convection-diffusion equation in perforated domains.
- The coarse scale discretization is based on a mixed formulation, which gives the mass conservation property.
- We presented numerical results to demonstrate a robustness of Generalized multiscale finite element method .
- Our results show that the presented method demonstrate accuracy for transport problem.
- In future we will consider construction of a multiscale basis functions with adding convection part to local problem.