Homogenization of the Steklov problem for elliptic equation

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Aperiodic case

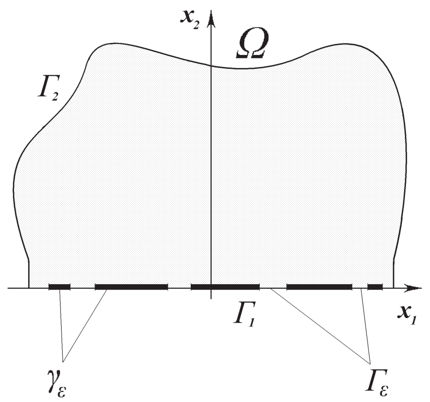
Assume that Ω is a domain in $\mathbb{R}^2$ lying in upper half–plane. Its boundary $\partial\Omega$ is piecewise smooth and consists of two parts: $\partial\Omega = \Gamma_1 \cup \Gamma_2$. The part Γ_1 is a segment $\left[-\frac{1}{2}; \frac{1}{2}\right]$ on the x-axis, while smooth part Γ_2 coincides with the straight lines $x_1 = -\frac{1}{2}$ and $x_1 = \frac{1}{2}$ in the neighborhood of points $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}, 0)$ respectively. Also assume that Γ_1 consists of alternating parts γ_ε^i, Γ_ε^i, and

$$\gamma_\varepsilon = \bigcup_{i=1}^{N_\varepsilon} \gamma_\varepsilon^i, \qquad \Gamma_\varepsilon = \bigcup_{i=1}^{N_\varepsilon} \Gamma_\varepsilon^i, \qquad \Gamma_1 = \gamma_\varepsilon \cup \Gamma_\varepsilon.$$

Suppose that for any i the following conditions are satisfied:

$$C^-\varepsilon \le |\Gamma_\varepsilon^i| \le C^+\varepsilon, \;\; C^-\varepsilon \le |\gamma_\varepsilon^i| \le C^+\varepsilon, \;\; \text{where} \;\; 0 < C^- < C^+ < +\infty.$$

Hereinafter ε is a positive small parameter.



Consider the following spectral problem of the Steklov type for the second order elliptic equation:

$$\begin{cases} L[u_\varepsilon] \equiv \displaystyle\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\left(a^{ij}(x)\frac{\partial u_\varepsilon}{\partial x_i}\right) = 0 \qquad \text{in} \qquad \Omega, \\ u_\varepsilon = 0 \qquad \text{on} \qquad \Gamma_2 \cup \Gamma_\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \vartheta} \equiv \displaystyle\sum_{i,j=1}^{2} a^{ij}(x)\frac{\partial u_\varepsilon}{\partial x_i}\nu_j = \lambda_\varepsilon u_\varepsilon \qquad \text{on} \qquad \gamma_\varepsilon. \end{cases} \tag{1}$$

Here $\nu = (\nu_1, \nu_2)^t$ is a unit outer normal to $\partial\Omega$. The coefficients $a^{ij}(x)$ are bounded measurable functions in Ω. The matrix $(a^{ij}(x))$ is positively definite, i.e.

$$\varkappa_1|\xi|^2 \le \sum_{i,j=1}^{2} a^{ij}(x)\xi_i\xi_j \le \varkappa_2|\xi|^2, \quad \text{where} \quad \varkappa_1 > 0, \varkappa_2 > 0.$$

Definition 1. *A function $u_\varepsilon \in W_2^1(\Omega, \Gamma_2 \cup \Gamma_\varepsilon) \setminus \{0\}$ is called the eigenfunction of the problem* (1) *corresponding to the eigenvalue λ_ε, if for any function $v \in W_2^1(\Omega, \Gamma_2 \cup \Gamma_\varepsilon)$ the following integral identity:*

$$\sum_{i,j=1}^{2} \int_{\Omega} a^{ij} \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx = \lambda_\varepsilon \int_{\gamma_\varepsilon} u_\varepsilon\, v\, ds. \tag{2}$$

holds true.

Theorem 1. *The first eigenvalue of the problem* (1) *is of order $\frac{1}{\varepsilon}$, i.e. it satisfies the following relation:*

$$\frac{K_1}{\varepsilon} \le \lambda_\varepsilon^1 \le \frac{K_2}{\varepsilon},$$

where K_1 and K_2 are positive constants. Moreover the first eigenfunction u_ε^1 converges in the norm $L_2(\Omega)$ and weakly converges in $W_2^1(\Omega)$ to zero.

Periodic case

Let now Ω be a domain in $\mathbb{R}^2$. $\partial\Omega$ is a simple smooth closed contour of the length 1. In the small neighborhood of $\partial\Omega$ the local coordinates (s, τ) are introduced.

Γ^ε — is an arbitrary non-empty closed one-dimensional set depending on $\varepsilon \in (0, 1]$ and lying in the interval $\Sigma = \{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 | 0 < \xi_1 < 1, \xi_2 = 0\}$.

It is assumed that $mes\, \Gamma^\varepsilon = O(\varepsilon)$.

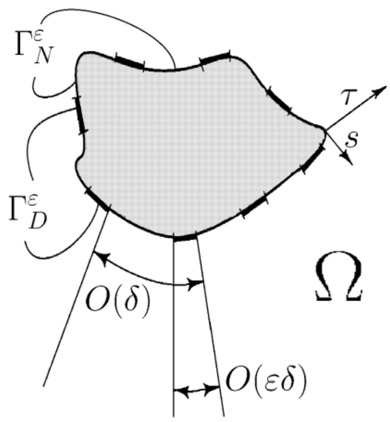
We write Γ_1^ε for the set formed by all integer shifts of Γ^ε along the axis $\xi_2 = 0$ and let Γ_D^ε be the image of Γ_1^ε under the mapping $s = \delta\xi_1, \tau = \delta\xi_2$. $\Gamma_N^\varepsilon = \partial\Omega \setminus \Gamma_D^\varepsilon$, $\varepsilon^{-1} \in \mathbb{N}$ for small ε; let δ depend on ε in such way that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

$B = \{\xi \in \mathbb{R}^2 |\, 0 < \xi_1 < 1, \xi_2 < 0\}$

Define the space $H_{1\text{-per}}(B, \Gamma^\varepsilon)$ as a completion with respect to the norm

$$\|v\|_1 = \Big(\iint_B |\nabla_\xi v|^2 d\xi + \int_\Sigma v^2 d\xi_1 \Big)^{\frac{1}{2}}$$

of the set of 1-periodic in ξ_1 functions in $C^\infty(\overline{B})$ which remain smooth after their 1-periodic extension with respect to ξ_1, vanish in a neighbourhood of Γ^ε and possess a finite Dirichlet integral over B.



Let

$$\theta_\varepsilon = \inf_{v \in H_{1\text{-per}}(B,\Gamma^\varepsilon)\setminus\{0\}} \frac{\iint\limits_B |\nabla_\xi v|^2 d\xi}{\int\limits_\Sigma v^2 d\xi_1},$$

We assume that there is a finite or infinite limit

$$\lim_{\varepsilon\to 0} \frac{\theta_\varepsilon}{\delta(\varepsilon)} = p \in [0,+\infty]. \tag{3}$$

Suppose that the condition (3) holds. Consider the following eigenvalue problems:

$$\begin{cases} \Delta u_\varepsilon^k = 0 & \text{in} \quad \Omega, \\ u_\varepsilon^k = 0 & \text{on} \quad \Gamma_D^\varepsilon, \\ \dfrac{\partial u_\varepsilon^k}{\partial \tau} = \lambda_\varepsilon^k u_\varepsilon^k & \text{on} \quad \Gamma_N^\varepsilon. \end{cases} \tag{4}$$

$$\begin{cases} \Delta u_0^k = 0 & \text{in} \quad \Omega, \\ u_0^k = 0 & \text{on} \quad \partial\Omega, \quad \text{as} \quad p = +\infty \\ \dfrac{\partial u_0^k}{\partial \tau} + p\, u_0^k = \lambda_0^k u_0^k & \text{on} \quad \partial\Omega, \quad \text{as} \quad p < +\infty. \end{cases} \tag{5}$$

Theorem 2. *1. Suppose that the condition* (3) *holds. Then there is a con-*

stant $K_3(k)$, independant of ε such that for sufficiently small ε

$$|\lambda_\varepsilon^k - \lambda_0^k| \leqslant K_3(k)\left(\sqrt{\theta_\varepsilon} + \left|\frac{\theta_\varepsilon}{\delta} - p\right|\right), \quad \text{if} \quad p < \infty,$$
$$\lambda_\varepsilon^k \to +\infty, \quad \text{if} \quad p = \infty.$$

2. Suppose that the multiplicity of the eigenvalue λ_0^{k+1} of the problem (5) *is equal to m: $\lambda_0^{k+1} = \cdots = \lambda_0^{k+m}$. Then for every eigenfunction of* (5) *with eigenvalue λ_0^{k+1} there is a linear combination $\overline{u}_\varepsilon$ of eigenfunctions $u_\varepsilon^{k+1}, \ldots, u_\varepsilon^{k+m}$ of the problem* (4) *with eigenvalues $\lambda_\varepsilon^{k+1}, \ldots, \lambda_\varepsilon^{k+m}$ respectively such that for sufficiently small ε*

$$\|\overline{u}_\varepsilon - u_0\|_{L_2(\Omega)} \leqslant K_4(k)\left(\sqrt{\theta_\varepsilon} + \left|\frac{\theta_\varepsilon}{\delta} - p\right|\right), \quad \text{if}\, p < \infty,$$

where the constant $K_4(k)$ does not depend on ε.

References

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[2] Chechkina A. Homogenization of spectral problems with singular perturbation of Steklov condition. *Izvestiya. Mathematics*, **81** (1, 2017), 199–236.