GMSDGM FOR SOLVING HEAT PROBLEM OF FREEZING AND THAWING PROCESS IN SOIL

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INTRODUCTION

In this work, we consider a numerical solution of a heat transfer problem with phase change in heterogeneous domains. For simulation of heat transfer processes with phase transitions, we use a classic Stefan model [1].
Computational implementation is based on generalized multiscale discontinuous Galerkin method (GMsDGM) [2, 3]. In this method the interior penalty discontinuous Galerkin method is used for the global coupling on a coarse grid. The main idea of these methods is to construct a small dimensional local solution space that can provide an efficient calculation on coarse grid level.
We present numerical results for different geometries to demonstrate an accuracy of the method.

[1] S. P. Stepanov, I. K. Sirditov, P. N. Vabishchevich, M. V. Vasilyeva, V. I. Vasilyev, and A. N. Tceeva. Numerical simulation of heat transfer of the pile foundations with permafrost. In *Numerical Analysis and Its Applications - 6th International Conference, NAA 2016, Lozenetz, Bulgaria, June 15-22, 2016, Revised Selected Papers*, pages 625–632, 2016.

[2] Y Efendiev and Thomas Y Hou. Multiscale finite element methods: theory and applications, 2009.

[3] E T Chung, Y Efendiev, M Vasilyeva, and Y Wang. A multiscale discontinuous galerkin method in perforated domains. In *PROCEEDINGS OF THE INSTITUTE OF MATHEMATICS AND MECHANICS*, volume 42, pages 212–229. INST MATHEMATICS & MECHANICS, NATL ACAD SCIENCES AZERBAIJAN 9 B VAHABZADEH STR, AZ1141, BAKU, 00000, AZERBAIJAN, 2016.

MATHEMATICAL MODEL

For simulation of heat transfer processes with phase transitions, we use a classic Stefan model

$$\left(\alpha(\phi) + \rho^+ L\phi'\right)\frac{\partial T}{\partial t} - \operatorname{div}\left(\lambda(\phi)\operatorname{grad} T\right) = 0,$$

where L is a specific heat of the phase transition, m is a porosity, ρ^+, c^+, λ^+ and ρ^-, c^-, λ^- are density, specific heat capacity and thermal conductivity of melted and frozen zones, respectively. We have the following coefficients

$$\alpha(\phi) = \rho^- c^- + \phi(\rho^+ c^+ - \rho^- c^-), \quad \lambda(\phi) = \lambda^- + \phi(\lambda^+ - \lambda^-),$$

$$c^-\rho^- = (1-m)c_{sc}\rho_{sc} + mc_i\rho_i, \quad \lambda^- = (1-m)\lambda_{sc} + m\lambda_i,$$

$$c^+\rho^+ = (1-m)c_{sc}\rho_{sc} + mc_w\rho_w, \quad \lambda^+ = (1-m)\lambda_{sc} + m\lambda_w.$$

and

$$\phi = \begin{cases} 0, & \text{when } T < T^*, \\ 1, & \text{when } T > T^*, \end{cases}.$$

The indexes sc, w, i denote the solid skeleton, water, and ice, respectively. In practice, the phase transitions occur in a small temperature range $[T^* - \Delta, T^* + \Delta]$. As the function ϕ we take ϕ_Δ:

$$\phi_\Delta = \begin{cases} 0, & T \le T^* - \Delta, \\ \dfrac{T - T^* + \Delta}{2\Delta}, & T^* - \Delta < T < T^* + \Delta, \\ 1, & T \ge T^* + \Delta, \end{cases}$$

FINE SCALE APPROXIMATION

The equation can be approximated using the IPDG finite element method while a standard implicit difference scheme can be used to approximate by time.
For each time layer, we have following variational formulation: find $T^{n+1} \in V$ $(n = 1, 2, ...)$ such that

$$m(T^{n+1}, v) + a_{DG}(T^{n+1}, v) = m(T^n, v), \quad \forall v \in V,$$

where

$$m(T^{n+1}, v) = \frac{1}{\tau}\int_\Omega \left(\alpha(\phi^n_\Delta) + \rho^+ L\phi'^n_\Delta\right) T^{n+1}\, v\, dx$$

$$a_{DG}(T^{n+1}, v) = \int_\Omega \left(\lambda(\phi^n_\Delta)\operatorname{grad} T^{n+1}, \operatorname{grad} v\right) dx -$$

$$- \int_\mathcal{E} \left(\langle\lambda(\phi^n_\Delta)\operatorname{grad} T^{n+1}\rangle, [v]n\right) ds -$$

$$- \int_\mathcal{E} \left(\langle\lambda(\phi^n_\Delta)\operatorname{grad} v\rangle, [T^{n+1}]n\right) ds - \frac{\gamma}{h}\int_\mathcal{E} \langle\lambda(\phi^n_\Delta)\rangle [T^{n+1}][v] ds,$$

and

$$m(T^n, v) = \frac{1}{\tau}\int_\Omega \left(\alpha(\phi^n_\Delta) + \rho^+ L\phi'^n_\Delta\right) T^n\, v\, dx.$$

where a linear basis functions are used for approximation.

DISCONTINUOUS GALERKIN GMSFEM

We show the main procedures of the problem solution with the use of GMsDGM:
We define coarse grid $\mathcal{T}_H$ in domain Ω, $\mathcal{T}_H = \cup_{i=1}^{N_c} K_i$ where K_i is coarse cell (local domain) and $\mathcal{T}_h$ is fine mesh with $H \ge h \ge 0$, where h is size of fine grid. A local subdomain in structured and unstructured coarse mesh are shown in Figure 5.

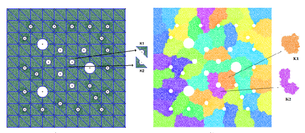

Figure 5: a) for structured grid (210 subdomains) b) for unstructured grid (50 subdomains)
We have follows steps:

1. Coarse grid generation $\mathcal{T}_H$;
2. Offline space construction;
   - Construction of snapshot space that will be used to compute an offline space.
   - Construction of a small dimensional offline space by performing dimension reduction in the space of local snapshots.
3. Solution of a coarse-grid problem for any force term and boundary condition.

DISCONTINUOUS GALERKIN GMSFEM

The offline computation we constructed local snapshot space $V_{snap}(K)$ for each local domain $K \in \mathcal{T}_H$. The snapshot space contains a large library of local basis functions. It has two types

$$V_{snap}(K) = V^b_{snap}(K) + V^i_{snap}(K).$$

Where $V^b_{snap}(K)$ in the coarse grid block K is defined as the linear span of all harmonic extensions, and the second local snapshot space $V^i_{snap}(K)$ is defines as $V^i_{snap}(K) = V^0_h(K)$, which is the limitations of V_h on K. For dimension reduction of the above snapshot spaces, we use eigenvalue problem and use eigenfunctions corresponding to the dominant eigenvalues as the multiscale basis functions.
Boundary basis. To construct the local snapshot space $V^b_{snap}(K)$ for each fine-grid node on the boundary of K, we find $\psi_{l,K} \in V_h(K)$ by solving

$$a_{DG}(\psi_{l,K}, v) = 0 \quad in \quad K$$
$$\psi_{l,K} = g_l \quad on \quad \partial K,$$

Interior basis. The local snapshot space $V^i_{snap}(K)$ for the coarse grid block K is defined as $V^i_{snap} = V^0_h(K)$. For the dimension reduction on the snapshot space, we use following spectral problem to identify the important modes: find $\phi^i_{m,K} \in V^0_h(K)$ satisfying

$$a_{DG}(\phi^i_{m,K}, v) = \nu(\phi^i_{m,K}, v) \quad in \quad V^0_h(K),$$
$$\phi^i_{m,K} = 0 \quad on \quad \partial K.$$

Solve coarse-scale system. Let V_H be the coarse-scale multiscale space, which consists of function are decided on each coarse grid block $K \in \mathcal{T}_H$.

$$V_H = span\{\phi^i_{k,j},\ i = 1, \dots, M_b,\ k = 1, \dots, M_i\ j = 1, \dots, N_c\},$$

where M_b is the number of boundary basis functions, M_i is the number of interior basis functions and N_c is the number of the local domains. The $\phi^i_{k,j}$ is continuous in coarse element K, but is not necessarily continuous along the coarse edges. We construct the coarse scale system in this offline space using discontinuous Galerkin coupling, where we need to penalize the jump of the solution on the coarse edges. We seek an approximation $T_H = \Sigma_{r,i} c_{r,i}\phi_{r,i}$ in V_H such that

$$a_{DG}(T_H, v) = (f, v), \forall v \in V_H.$$

RESULTS

Case 1

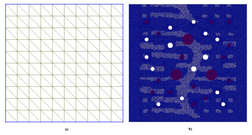

Figure 1: a) Coarse grid with 210 elements. b) Fine grid with 30078 elements. Blue color: first subdomain. Grey color: second subdomain. Red color: third subdomain.

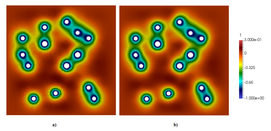

Figure 2: a) Solution on a coarse grid using 20 boundary and 10 interior multiscale basis functions (DOF_c = 6300, 6.9% from DOF_f). b) Solution on a fine grid (DOF_f = 90234).

	$\|\|e\|\|_{L^2}$ $M_b = 10$	$\|\|e\|\|_a$ $M_b = 10$	$\|\|e\|\|_{L^2}$ $M_b = 20$	$\|\|e\|\|_a$ $M_b = 20$	$\|\|e\|\|_{L^2}$ $M_b = 30$	$\|\|e\|\|_a$ $M_b = 30$
$M_i = 0$	-	-	17.41	34.38	17.13	31.58
$M_i = 10$	10.92	37.59	7.16	19.27	-	-

Table 1: Solution errors for different number of multiscale basis functions

Case 2

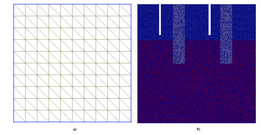

Figure 3: a) Coarse grid with 210 elements. b) Fine grid with 26116 elements. Red color: first subdomain. Blue color: second subdomain. Grey color: third subdomain.

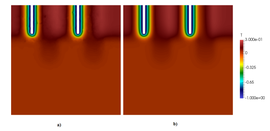

Figure 4: Solution on a coarse grid using 20 boundary and 10 interior multiscale basis functions (DOF_c = 6300, 8.0% from DOF_f). b) Solution on a fine grid (DOF_f = 78348).

	$\|\|e\|\|_{L^2}$ $M_b = 10$	$\|\|e\|\|_a$ $M_b = 10$	$\|\|e\|\|_{L^2}$ $M_b = 20$	$\|\|e\|\|_a$ $M_b = 20$	$\|\|e\|\|_{L^2}$ $M_b = 30$	$\|\|e\|\|_a$ $M_b = 30$
$M_i = 0$	-	-	17.41	34.38	17.13	31.58
$M_i = 10$	10.92	37.59	7.16	19.27	-	-

Table 2: Solution errors for different number of multiscale basis functions.

For numerical comparision, we consider the relative L^2 and energy errors

$$||e||_{L^2} = \frac{\sqrt{\int_\Omega (T_h - T_H)^2 dx}}{\sqrt{\int_\Omega T_h^2 dx}}, \quad ||e||_a = \frac{\sqrt{a_{DG}(T_h - T_H, T_h - T_H)}}{\sqrt{a_{DG}(T_h, T_h)}}$$

where $e = T_h - T_H$, T_h and T_H are the fine-scale and multiscale solutions, respectively.