Numerical Solution Of Unsteady Problems With Different Time Scales

MULTISCALE MODEL REDUCTION AND HIGH PERFORMANCE COMPUTING LABORATORY

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Introduction

Many of the phenomena and objects studied are characterized by the multiscale processes. When considering dynamic processes, the principle is multiscale time, when the processes go with different speeds [1, 2].
There is a special difference scheme [3], which taking into account the behavior of the solution. This approach is known as the heterogeneous multiscale method (HMM) [4, 5]. In the work [6], the dynamic multiscale processes is suggested to be taken into account using iterative splitting schemes.
The present paper is devoted to the construction of inhomogeneous approximations by time for problems with fast and slow components of the solution on the basis of splitting schemes. Heterogeneous schemes are most naturally associated with schemes of componentwise splitting [7, 8], when the operator of the problem describing fast processes splits into the sum of one-type operators.

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Problem statement

We consider the Cauchy problem for a homogeneous linear equation:

$$\frac{du}{dt} + Au = 0, \quad 0 < t \le T, \qquad (1)$$

$$u(0) = u^0. \qquad (2)$$

We assume that the operator A is constant and nonnegative

$$\frac{d}{dt}A = A\frac{d}{dt}, \quad A \ge 0$$

in a finite-dimensional Hilbert space H. There is an estimate of stability from the initial data:

$$\|u(t)\| \le \|u_0\|. \qquad (3)$$

We assume that the equation (1) describes a two-scale dynamic process and we select two terms:

$$A = S + F, \qquad (4)$$

$$\frac{d}{dt}S = S\frac{d}{dt}, \quad S \ge 0, \quad \frac{d}{dt}F = F\frac{d}{dt}, \quad F \ge 0.$$

The operator S and F describes slow and fast processes.

Slow and fast processes

Slow processes is modeled with a uniform time step τ:

$$t^n = n\tau, \quad n = 0, 1, ..., N, \quad \tau N = T,$$

and using a standard scheme with weights

$$\frac{v^{n+1} - v^n}{\tau} + S\left(\sigma_S v^{n+1} + (1-\sigma_S)v^n\right) = 0. \qquad (5)$$

Lemma 1. *If*

$$(Sv, v) + (\sigma_S - 0.5)\tau\|Sv\|^2 \ge 0, \qquad (6)$$

the scheme (5) is stable and the solution on the layer is satisfied:

$$\|v^{n+1}\| \le \|v^n\|. \qquad (7)$$

For fast processes, the transition from the t^n layer to the t^{n+1} layer is performed using the step τ/m for m steps:

$$\frac{v^{n+\frac{\alpha}{m}} - v^{n+\frac{\alpha-1}{m}}}{\tau/m} + F\left(\sigma_F v^{n+\frac{\alpha}{m}} + (1-\sigma_F)v^{n+\frac{\alpha-1}{m}}\right) = 0. \qquad (8)$$

Lemma 2. *If*

$$(Fv, v) + (\sigma_F - 0.5)\frac{\tau}{m}\|Fv\|^2 \ge 0, \qquad (9)$$

the scheme (8) is stable and the estimation (7) is satisfied.

Componentwise splitting schemes

For the operator A in the equation (1), we set

$$A = \sum_{\alpha=1}^{m+1} A_\alpha, \; A_\alpha = \frac{1}{m}F, \; \alpha = 1, 2, ..., m, \; A_{m+1} = S. \qquad (10)$$

In this case we have the same time scale (time step τ) for all operators $A_\alpha, \alpha = 1, 2, ..., m+1$.
For numerical solution of the problem (1), (2), (10) the different splitting schemes [7, 8] can be used.
An approximate solution of the problem (1), (2), (10) and the transition from the time layer t^n to the layer t^{n+1} is determined from conditions

$$\frac{y^{n+\frac{\alpha}{m+1}} - y^{n+\frac{\alpha-1}{m+1}}}{\tau} + A_\alpha\left(\sigma_\alpha y^{n+\frac{\alpha}{m+1}} + (1-\sigma_\alpha)y^{n+\frac{\alpha-1}{m+1}}\right) = 0,$$

$$\alpha = 1, 2, ..., m+1, \quad n = 0, 1, ..., N-1, \qquad (11)$$

$$\sigma_\alpha = \sigma_F, \quad \alpha = 1, 2, ..., m, \quad \sigma_{m+1} = \sigma_S.$$

Theorem 1. *Under the constraints (6), (9), the componentwise splitting scheme for multiscale scheme (11) is stable and the solution on the layer is satisfies*

$$\|y^{n+1}\| \le \|y^n\|, \quad n = 0, 1, ..., N-1. \qquad (12)$$

The proof is based on Lemmas 1 and 2.

Vector additive schemes

Instead of finding the scalar function u, we look for the vector $\boldsymbol{u} = \{u_1, u_2, \ldots, u_{m+1}\}$. Each individual component is defined as the solution of the same problem

$$\frac{du_\alpha}{dt} + \sum_{\beta=1}^{m+1} A_\beta u_\beta = 0, \quad \alpha = 1, 2, ..., m+1, \quad 0 < t \le T, \quad (14)$$

$$u_\alpha(0) = u^0, \quad \alpha = 1, 2, ..., m+1. \qquad (15)$$

In this case, $u_\alpha(t) = u(t)$, $\alpha = 1, 2, ..., m+1$, and therefore, as a solution of the original problem (1), (2), we can take any component of the vector $\boldsymbol{u}(t)$.
For the approximate solution of the problem (14), (15) we use a two-layer vector additive scheme

$$\frac{y_\alpha^{n+1} - y_\alpha^n}{\tau} + \sum_{\beta=1}^{\alpha} A_\beta y_\beta^{n+1} + \sum_{\beta=\alpha+1}^{m+1} A_\beta y_\beta^n = 0, \qquad (16)$$

$$\alpha = 1, 2, ..., m+1, \quad n = 0, 1, ..., N-1,$$

$$y_\alpha^0 = u^0, \quad \alpha = 1, 2, ..., m+1. \qquad (17)$$

Theorem 2. *The vector additive scheme (10), (16), (17) is unconditionally stable and the estimate for the solution is performed*

$$\|y_\alpha^{n+1}\| \le \|y_\alpha^n\| + \tau\|Au^0\|, \quad \alpha = 1, 2, ..., m+1, \quad n = 0, 1, ..., N-1. \qquad (18)$$

Convection-diffusion problem

We consider the Cauchy problem in the unit square Ω

$$\frac{dw}{dt} + \mathcal{F}w + \mathcal{S}w = 0, \quad w(0) = w^0. \qquad (13)$$

Diffusive transfer corresponds to the slow operator

$$\mathcal{S}w = -\frac{1}{\mathrm{Pe}}\triangle w, \quad \boldsymbol{x} \in \Omega,$$

for w satisfying the boundary conditions $\frac{\partial w}{\partial \nu} = 0$, $\boldsymbol{x} \in \partial\Omega$, where ν is the boundary normal, Pe is the Peclet number.
Convective transfer corresponds to the fast operator (symmetrical form):

$$\mathcal{F}w = \frac{1}{2}\mathrm{div}(\boldsymbol{v}\, w) + \frac{1}{2}\boldsymbol{v} \cdot \mathrm{grad}\, w.$$

The convective transfer is determined by the velocity of the medium $\boldsymbol{v}(\boldsymbol{x})$, and the non-flow condition $\boldsymbol{v} \cdot \boldsymbol{\nu} = 0$, $\boldsymbol{x} \in \partial\Omega$.
For the operators $\mathcal{S}$ and $\mathcal{F}$, we have

$$\mathcal{S} = \mathcal{S}^* \ge 0, \quad \mathcal{F} = -\mathcal{F}^*.$$

We take the initial condition w^0 and the velocity $\boldsymbol{v} = (v_1, v_2)$

$$w^0(\boldsymbol{x}) = x_1^2(1-x_1)^4 x_2^2(1-x_2)^4,$$

$$v_1 = \frac{\partial \psi}{\partial x_2}, \quad v_2 = -\frac{\partial \psi}{\partial x_1}, \quad \psi(\boldsymbol{x}) = \frac{1}{\pi}\sin(\pi x_1)\sin(\pi x_2).$$

Numerical experiments

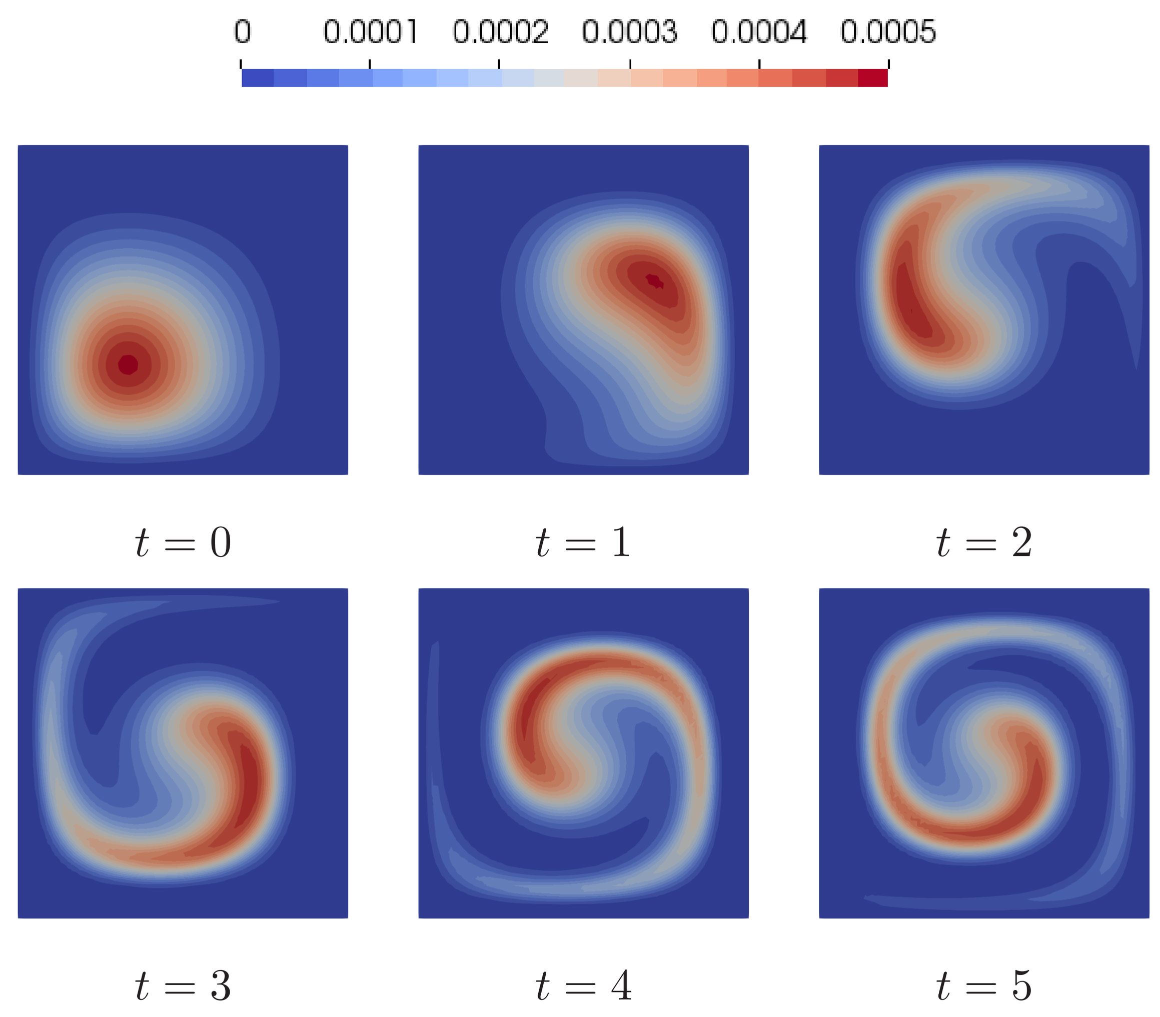

Figure 1: The solution of the problem for Pe = 10^4.

The relative error of the approximate solution is

$$\varepsilon(t) = \frac{\|u - \bar{u}\|}{\|\bar{u}\|},$$

where $\bar{u}$ is the reference solution obtained using a symmetric scheme (Crank-Nicolson scheme) with $\tau = 1.25 \cdot 10^{-3}$.
Fig. 2, 3 shows the accuracy of the implicit and inhomogeneous schemes at various time steps and number of fractional steps.

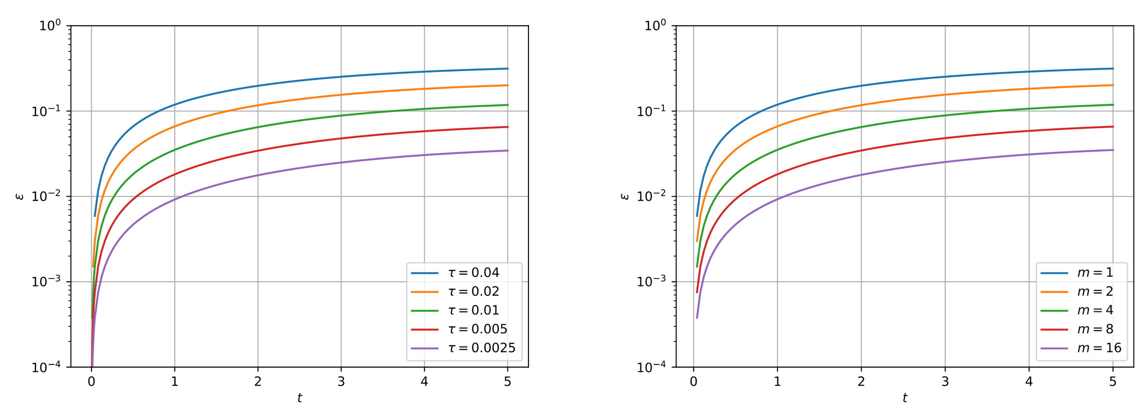

Figure 2: Implicit and inhomogeneous schemes: Pe = 10^4.

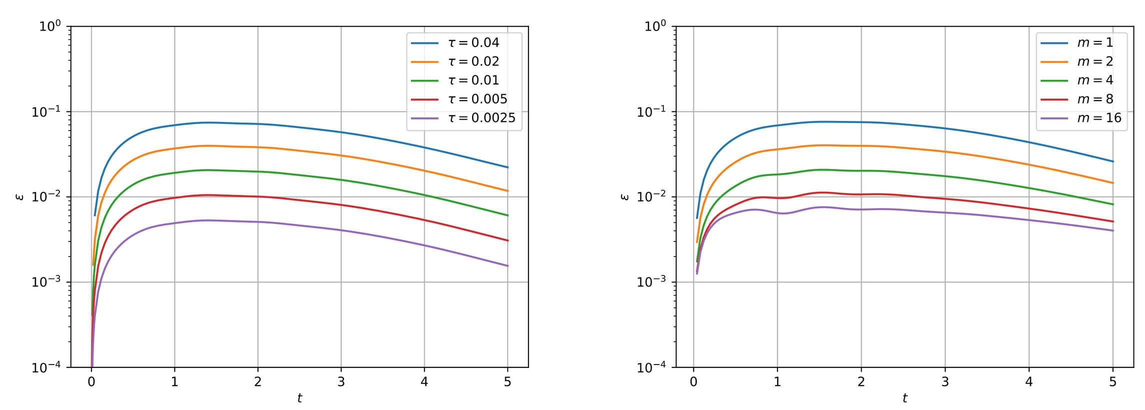

Figure 3: Implicit and inhomogeneous schemes: Pe = 10^2.

When using a detailed grid for convective transfer, the accuracy increasing of the approximate solution is observed. At the same time, the accuracy for large Peclet numbers remains approximately the same as without using splitting for a detailed grid.