GMsFEM FOR POROELASTICITY PROBLEMS IN HETEROGENEOUS MEDIA

{ Alexey Tyrylgin & Maria Vasilyeva }   North-Eastern Federal University, Yakutsk, Russia

MULTISCALE MODEL REDUCTION AND HIGH PERFORMANCE COMPUTING LABORATORY

Introduction

In this work, we consider the poroelasticity problems in heterogeneous porous media. Mathematical model contains coupled system of the equations for pressure and displacements. For the numerical solution we research and implement a Generalized Multiscale Finite Element Method(GMsFEM). This method solves a problem on a coarse grid by creation of the local multiscale basic functions.
We compare the solutions by choosing different numbers of multiscale basis functions and results show that GMsFEM can provide good accuracy.

Mathematical model

We consider linear poroelasticity problem where we wish to find a pressure p and displacements u satisfying

$$-\operatorname{div}\sigma(u) + \alpha\operatorname{grad}(p) = f_u(x,t),\ x\in\Omega,\ 0<t\le T, \tag{1}$$

$$\alpha\frac{\partial\operatorname{div}u}{\partial t} + \frac{1}{M}\frac{\partial p}{\partial t} - \operatorname{div}\left(\frac{k}{v}\operatorname{grad}p\right) = f_p(x,t),\ x\in\Omega,\ 0<t\le T, \tag{2}$$

Here the primary sources of the heterogeneities in the physical properties arise from elasticity tensor in stress σ and permeability k. We denote M to be the Biot modulus, ν is the fluid viscosity, σ is the linear stress, ε is the strain tensor and α is the Biot-Willis fluid-solid coupling coefficient. σ and ε are given as

$$\sigma(u) = 2\mu\varepsilon(u) + \lambda\operatorname{div}(u)I,$$

$$\varepsilon(u) = \frac{1}{2}(\operatorname{grad}u + \operatorname{grad}u^T),$$

where μ,λ are Lame coefficients, I is the identity tensor. In the case where the media has heterogeneous material properties the coefficients μ and λ may be highly variable.
The problem for the system of equations (1) - (2) is considered in a bounded domain Ω.
For displacements on the boundary $\partial\Omega = \Gamma_D + \Gamma_N$ we set

$$u = 0,\ x\in\Gamma_D,\ -\sigma n = 0,\ x\in\Gamma_N, \tag{3}$$

Where n is the unit normal to the boundary. Similarly, for pressure, we set Dirichlet and Neuman boundary conditions

$$p = 0,\ x\in\Gamma_3\cup\Gamma_4\cup\Gamma_5 \tag{4}$$

At the initial time t, the following condition is given

$$\varepsilon(x,0) = 0,\ p(x,0) = p_0,\ x\in\Omega, \tag{5}$$

for deformations and pressure.

Space approximation

For approximation by space, we use finite element method abd write the variational formulation for equations (1) - (2) with the initial boundary conditions (3) - (5): find $u\in W, p\in V$ such that

$$a(u,v) + g(p,v) = l^u(v),\ v\in W'$$

$$d(\dot{u},q) + b(p,q) = l^p(q),\ q\in V'.$$

Here the bilinear and linear forms are given as follows:

$$a(u,v) = \int_\Omega \sigma(u)\varepsilon(v)dx,\ g(p,v) = \alpha\int_\Omega \operatorname{grad}p\, v dx,$$

$$d(\dot{u},q) = \alpha\int_\Omega \frac{\partial\operatorname{div}u}{dt}qdx,\ c(p,q) = \frac{1}{M}\int_\Omega pqdx,$$

$$b(p,q) = \int_\Omega (k\operatorname{grad}p, \operatorname{grad}q)dx,$$

$$l^u(v) = \int_\Omega f_u v dx,\ l^p(q) = \int_\Omega f_p q dx + \int_{\partial\Omega} p_1 q ds.$$

Bilinear forms $a(\cdot,\cdot)$,$b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are symmetric and positively defined, $d(v,v) = -g(v,v)$ for $v\in V, v\in W$.
Approximation by time
For approximation by time, we use an implicit difference scheme

$$\int_\Omega \sigma^e(u^{n+1})\varepsilon(v)dx + \alpha\int_\Omega \operatorname{grad}p^n v dx = f_u(x,t)$$

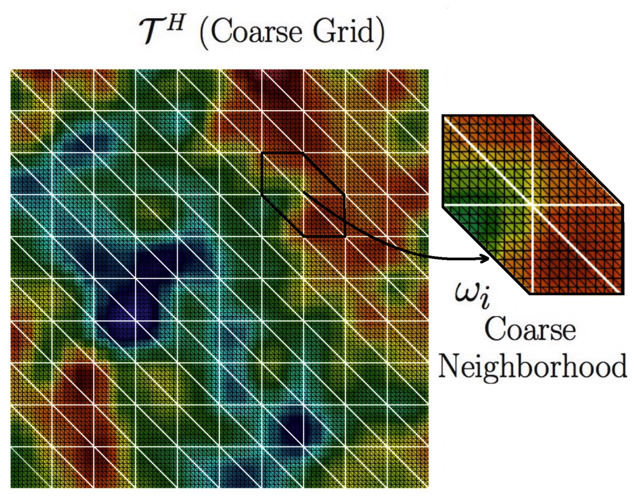
$$\alpha\int_\Omega \frac{\operatorname{div}(u^{n+1}-u^n)}{\tau}qdx + \frac{1}{M}\int_\Omega p^{n+1}qdx +$$

$$+\int_\Omega (k\operatorname{grad}p^{n+1}, \operatorname{grad}q)dx = f_p(x,t),$$

where we use linear basis functions for pressure and for displacements.

GMsFEM for poroelasticity

In this section, we describe in detail the Generalized Multiscale Finite Element Method (GMsFEM) for solution of the poroelasticity problems in heterogeneous media. GMsFEM contains three steps: (1) Construction of the coarse and fine meshes and local domains where we construct multiscale basis functions, (2) Solve local spectral problems for construction of the multiscale basis functions and (3) Construction and solution of the coarse scale approximation on multiscale space.
Step 1. Construction of the coarse and fine meshes ($\mathcal{T}_H$ and $\mathcal{T}_h$) and local domains (ω_i)



Step 2. *Spectral problem.* For construction of the multiscale basis functions. we solve a local spectral problems in domain ω_i for displacement and pressure separately. The local spectral problems

Pressure	Displacements
$a^u(u,v) = \lambda_u, s^u(u,v),$	$a^p(p,q) = \lambda_p s^p(p,q),$
$a^u(u,v) = \int_{\omega_i}\sigma(u)\varepsilon(v)dx,\ s^u = \int_{\omega_i}(\lambda+\mu)(u,v)dx,$	$a^p(p,q) = \int_{\omega_i} k\operatorname{grad}p, \operatorname{grad}q\, dx,\ s^p = \int_{\omega_i}(kp,q)dx,$

where v, q is a trial functions. A multiscale spaces V_h, Q_h, we will form using eigenvectors $\varphi_1, \varphi_2, ...\varphi_L, \psi_1, \psi_2, ...\psi_L$ corresponding to the first smallest L eigenvalues, where $\lambda_1 \le \lambda_2 \le ... \le \lambda_L$.
We can write spectral problems in the matrix form

Pressure	Displacements
$A^u\varphi = \lambda_u S^u\varphi, \quad x\in\omega_i,$	$A^p\psi = \lambda_p S^p\psi, \quad x\in\omega_i,$

where A^u, S^u matrices for displacements and A^p, S^p for pressure.
In the fact that the spectral problem must be solved many times. Therefore, it is possible to reduce the dimension of the problem. The method consists in solving the spectral problem only at the boundary nodes of the domain ω_i. That is, we solve problems (1) in another spaces, this spaces is called a snapshot space, denoted as V_{snap} for displacement and Q_{snap} for pressure. To go into spaces V_{snap}, Q_{snap}, you need to obtain a transition matrices R^u_{snap}, R^p_{snap}.

Pressure	Displacements
$R^u_{snap} = [\varphi_1^{snap}, ..., \varphi_{L_i}^{snap}],$	$R^p_{snap} = [\psi_1^{snap}, ..., \psi_{L_i}^{snap}],$

Next, we move to spaces V_{snap}, Q_{snap} and obtain the following eigenvalue problem

Pressure	Displacements
$\overline{A}^u\overline{\varphi} = \lambda_u\overline{S}^u\overline{\varphi}, \quad x\in V_{snap},$	$\overline{A}^p\overline{\psi} = \lambda_p\overline{S}^p\overline{\psi}, \quad x\in Q_{snap}.$

Pressure	Displacements
$\overline{A}^u = R^u_{snap}A^u(R^u_{snap})^T,$	$\overline{S}^u = R^u_{snap}S^u(R^u_{snap})^T,$
$\overline{A}^u = R^u_{snap}A^u(R^u_{snap})^T,$	$\overline{S}^p = R^p_{snap}S^p(R^p_{snap})^T.$

where $\varphi_j = (R^u_{snap})^T\overline{\varphi}$ and $\psi_j = (R^p_{snap})^T\overline{\psi}$. As a result of solving the spectral problem, we have obtained the bases that are used to move from small scale to coarse. Bases can be different numbers, with the increase in the number of bases, the accuracy of the solution will improve.
For obtaining conforming basis functions we use linear partition of unity functions.

Step 3. *Assembling a matrices in a multiscale space*
Next, we construct transition matrix R from a fine grid to a coarse grid and use it for reducing the dimension of the problem. The transition matrix

$$R = \begin{pmatrix} R_u \\ R_p \end{pmatrix} = \begin{pmatrix}
0 & \cdots & B_{1_x}*P_1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & B_{1_y}*P_1 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & \cdots & B_{1_p}*P_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & B_{L_x}*P_1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & B_{L_y}*P_1 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & \cdots & B_{L_p}*P_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & B_{L_x}*P_{N_c} & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & B_{L_y}*P_{N_c} & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & \cdots & B_{L_p}*P_{N_c} & \cdots & 0
\end{pmatrix},$$

where subindexes x, y, p denote the displacement in x, y directions and the pressure, respectively. B_j and P_i are multiscale basis functions and linear partition of unity functions, $j = 0...L$, $i = 0...3*N_c$, L is the number of bases, N_c is the number of vertices of a coarse grid, N_f id the number of vertices of a fine grid. Thus, the dimension of the matrix R is equal to $3*N_cL \times 3*N_f$.
Then the system of equations can be translated into a coarse grid

$$A_cu_c + G_cp_c = F_c,$$
$$D_cu_c + B_cp_c = Y_c,$$

where $A_c = R_uA_fR_u^T, B_c = R_pB_fR_p^T, F_c = R_uF_f, Y_c = R_pY_f, G_c = R_uG_fR_p^T, D_c = R_pD_fR_u^T$ and

$$A_fu = \int_\Omega \sigma(u)\varepsilon(v)dx, \quad B_fp = \frac{1}{M}\int_\Omega \frac{p}{\tau}qdx + \int_\Omega (k\operatorname{grad}p, \operatorname{grad}q)dx,$$

$$G_fp = \int_\Omega \alpha\operatorname{grad}(p)\cdot v\, dx, \quad D_fu = \alpha\int_\Omega \frac{\operatorname{div}u}{\tau}qdx, \quad F_f = \int_\Omega f_uvdx, \quad Y_f = \int_\Omega f_pqdx + \int_{\partial\Omega} p_1qds$$

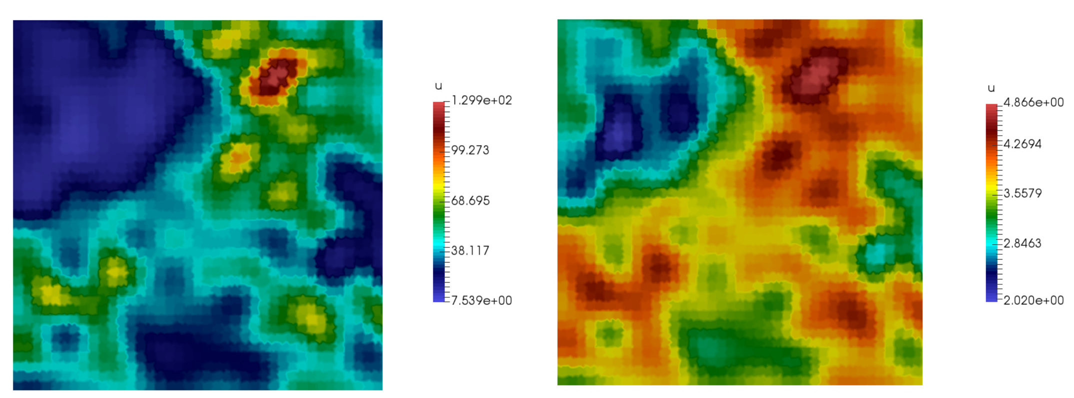
after obtaining of a coarse-scale solution, we can reconstruct fine-scale solition

$$u_{ms} = R_u^Tu_c, \quad p_{ms} = R_p^Tp_c$$

As a result, we obtain a solution on a fine grid for the problem of poroelasticity.

Results

In this section, we present numerical examples to demonstrate the performance of the GMsFEM for computing the solution of the poroelasticity problem in heterogenous domains. For the Biot modulus, we take $M = 62500000$ and set $\alpha = 1.0$, the Poisson's ratio is $\nu = 0.3$. Permeability field and elastic modulus are shown in figure above. When solving of poroelasticity problem the number of basis functions for displacement and pressure is the same. Time parameters: $T_{max} = 0.001$ and $dt = 0.0001$.

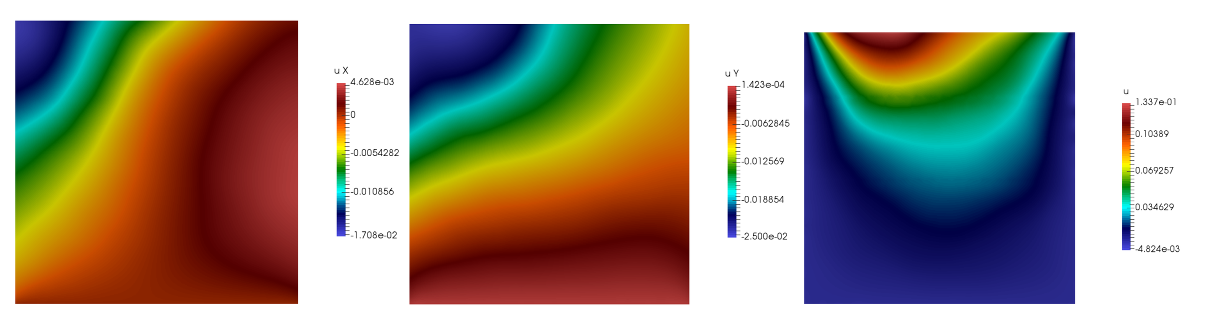


Heterogeneous permeability and elasticity parameter E.

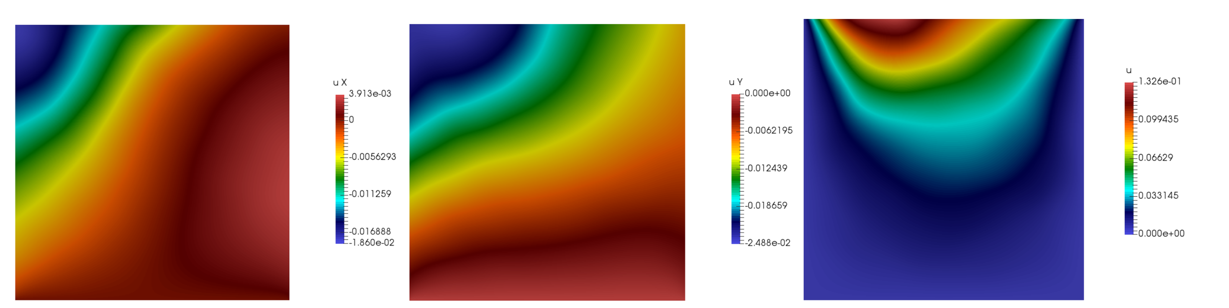
Basis	N_c^u	N_c^p	Displacement errors L2	Displacement errors H1
1	242	121	16.4053	22.4455
2	484	242	6.76872	12.4989
4	968	484	7.67148	9.8243
8	1936	968	6.92451	5.4054
12	2904	1936	2.06456	3.64363

Basis	Pressure errors L2	Pressure errors H1
1	48.1185	36.2861
2	21.6706	24.9166
4	19.0567	17.9077
8	3.48599	13.9048
12	1.4701	11.8512

Numerical comparision of the solutions by choosing different numbers of multiscale basis functions in GMsFEM.



Distribution of displacement for X and Y directions and pressure at the final time for **GMsFEM**.



Distribution of displacement for X and Y directions and pressure at the final time on **fine grid**.

Conclusion

In this work, a Generalized Multiscale Finite Element Methods for the poroelasticity problem in heterogeneous porous media is applied. For the numerical solution of the arising coupled system of equations for pressure and displacements, an approximation of the problem is constructed usingfinite element method with linear basis functions. We show the errors relative with varying multiscale basis functions and distribution of displacement along X and Y direction and pressure at the final time.

References

[1] Alexandr E Kolesov, Petr N Vabishchevich, and Maria V Vasilyeva. Splitting schemes for poroelasticity and thermoelasticity problems. *Computers & Mathematics with Applications*, 67(12):2185–2198, 2014.

[2] Donald L Brown and Maria Vasilyeva. A generalized multiscale finite element method for poroelasticity problems i: linear problems. *Journal of Computational and Applied Mathematics*, 294:372–388, 2016.

[3] Yalchin Efendiev and Thomas Y Hou. *Multiscale finite element methods: theory and applications*, volume 4. Springer Science & Business Media, 2009.

[4] Yalchin Efendiev, Juan Galvis, and Thomas Y Hou. Generalized multiscale finite element methods (gmsfem). *Journal of Computational Physics*, 251:116–135, 2013.

Contact Information

Email koc9tk@mail.ru