Not improvable in the order of accuracy a posteriori error bounds for approximate solutions of reaction-diffusion equations

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The report is dedicated to the error control of approximate solutions of reaction-diffusion equations which are represented by the equation $-\Delta u + \sigma u = f$ in $\Omega \subset R^m$, $m \geq 1$, with $0 \leq \sigma = \text{const}$ and the Dirichlet doundary condition $u = 0$ on the boundary $\partial\Omega$. Sructure of many modern a posteriori error majorants traces back to the majorant of Aubin (1972). For the square of the energy norm of the error $e = u - v$, this majorant contains the summand $\sigma^{-1}\|r\|^2_{L^2(\Omega)}$, where $r = f - \sigma v + \nabla \cdot \mathbf{z}$, v is the approximation of the exact solution u of the problem, satisfying Dirichlet boundary condition, and $\mathbf{z}$ is an arbitrary sufficiently smooth testing flax field. Therefore, the accuracy of the majorant deteriorates when σ is close to the point $\sigma = 0$, where the majorant becomes senseless. It can be shown that the range of "bad" σ is not small, and for f.e.m. (finite element method) solutions on quasiunform discretizations of size h it covers $[0, ch^{-2})$, $c = \text{const}$. Several attempts were made recently to improve Aubin's majorant; besides for $\sigma = 0$ Frolov and Repin (2002) presented the majorant with the pointed out term replaced by $2c_\Omega\|f + \nabla \cdot \mathbf{z}\|^2_{L^2(\Omega)}$, where c_Ω is the constant from the Friedrichs inequality. Terms like the ones mentioned above are present in the majorants of numerous papers. The common feature of such a posteriory error majorants is that they are not consistent with the corresponding a priori error bounds. For $\mathbf{z}$ having the same order of accuracy with the flax $\mathbf{z}_{\text{fem}}$, deined by f.e.m. solutions, the energy norm of the error can be overestimated in ch^{-1} times and more. In the literature there are known two remedies to cope with this drawback. One is applying the minimization procedure with respect to $\mathbf{z}$ to the right part of the bound, another is based on the use of $\mathbf{z}$, obtained by the smoothing and equilibration procedures applied to $\mathbf{z}_{\text{fem}}$. The second way can be sufficiently cheap (since smoothing and equilibration can be arranged patch wise) and accurate, as illustrated, *e.g.*, by the algorithm for f.e.m. with simplicial linear finite elements presented in the paper of Ainsworth and Vejchodsky (2015). At the same time efficient algorithms of eqilibration without compromising the accuracy of $\mathbf{z}_{\text{fem}}$ became much more complex for higher order f.e.m.

We use a new way of derivation of a posteriori error bounds which results in the bounds well defined for all $\sigma \geq 0$ and consistent with the a priori error bounds. In particular, for f.e.m. solutions we obtain the majorant, which continuously depends on σ, for $\sigma \geq 1/(c_\dagger h^2)$ coincides with the Aubin's majorant, is consistent with the not improvable a priori error bounds and, therefore, itself is not improvable. The main difference between the old and new majorants is in the multiplers before L^2-norms of residuals. It is proved that in fact for $\sigma \in [0, 1/(c_\dagger h^2)]$ the summand $\sigma^{-1}\|r\|^2_{L^2(\Omega)}$ in Aubin's majorant should be replaced by $2\theta c_\dagger h^2\|r\|^2_{L^2(\Omega)}$, where $1/2 \leq \theta = 1/(1 + c_\dagger h^2\sigma) \leq 1$ on the pointed out segment. This is not only greatly improves

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accuracy, but also simplifies obtaining the testing flax $\mathbf{z}$, because the minimization and equilibration become unnecessary. It is sufficient to smooth the f.e.m. flax $\mathbf{z}_{\text{fem}}$ and obtain such $\mathbf{z}$ that $\mathbf{z} \in [L^2(\Omega)]^m$, $\operatorname{div} \mathbf{z} \in L^2(\Omega)$ with retention of the orders of accuracy of the f.e.m. flax.

Accuracy and simplicity of evaluation of constants entering the majorant is crucially important for applications. With the purpose to provide these properties and to minimize constants, we use several ways of derivation of a posteriori error majorants. In the most general majorant, the constants are entirely depend on σ_* from the inequality $|e|_1^2/\|e\|_0^2 \le \sigma_*$, which for f.e.m. error is equivalent to the inequality $\|e\|_0^2 \le c_\dagger h^2 |e|_1^2$, proved, as it is known, with the use of Aubin-Nitsche trick. Additionally we prove (this result was obtained jointly with V. Kostylev) that for f.e.m. error majorants the constant $c_\dagger$ can be replaced by the constant $\hat{c}$ from the estimate $\|v - \mathcal{I}_h v\|_0^2 \le \hat{c} h^2 |v - \mathcal{I}_h v|_1^2$, $\forall v \in H^1(\Omega)$, where $\mathcal{I}_h$ is the quasi-interpolation operator of Skott and Zhang (1990) for obtaining approximations of functions from $H^1(\Omega)$ by means of functions from the finite element subspace of continuous piece wise linear functions. There are some other ways of defining the needed constants as well.

A brief presentation of a part of the results delivered in this talk can be found in the papers of Korneev [3, 4]. In part they are generalized upon $2n^{th}$ order elliptic equations in [5].

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